

U.G. 5th Semester Examination - 2024

MATHEMATICS

[HONOURS]

Discipline Specific Elective (DSE)

Course Code : MATH-H-DSE-T-1B

(Partial Differential Equation & Laplace Transforms)

[New Syllabus]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

GROUP-A

[Marks : 20]

1. Answer any ten questions: $2 \times 10 = 20$
- a) Form a partial differential equation by eliminating m and n from $z = (x^2 + m)(y^2 + n)$.
 - b) Obtain the partial differential equation of all spheres of radius a having centres in the xy plane.
 - c) Find a complete integral of $x(1 + y)p = y(1 + x)q$.

[Turn over]

- d) Define complete integral, general integral and singular integral of a first order partial differential equation $F(x, y, z, p, q) = 0$.
- e) State Cauchy-Kowalewski theorem.
- f) Interpret the equation $Pp + Qq = R$ and $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ geometrically and establish the relationship between the two.
- g) Show that the equation $(1 - x^2)u_{xx} + 2yu_{xy} + u_{yy} = 0$ is hyperbolic for all (x, y) outside the region bounded by the circle $x^2 + y^2 = 1$, parabolic on the boundary of this region and elliptic for all (x, y) inside this region.
- h) Show that the function $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ satisfies the Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.
- i) Show that there always exists an integrating factor for a Pfaffian differential equation in two variables.
- j) Explain the order and degree of a PDE with examples.

- k) Prove that the direction cosines of the tangent at the point (x, y, z) to the conic $ax^2 + by^2 + cz^2 = 1$, $x + y + z = 1$ are proportional to $(by - cz, cz - ax, ax - by)$.
- l) From definition, find the Laplace transform of $\sinh at$.
- m) If $L^{-1}\{F(s)\} = f(t)$, then show that $L^{-1}\{F(s - a)\} = e^{at}L^{-1}\{F(s)\}$.
- n) Using the shifting property in Laplace transforms, find $L\{e^{-3t}t^3\}$.
- o) Find $L^{-1}\left\{\frac{s+3}{s^2-4s+20}\right\}$.

GROUP-B

[Marks : 20]

2. Answer any four questions: 5×4=20
- a) Find the surface which intersects the surfaces of the system $z(x + y) = c(3z + 1)$ orthogonally and which passes through the circle $x^2 + y^2 = 1$, $z = 1$.
- b) Obtain the complete integral of $z^2(p^2 + q^2) = 1$ using Charpit's method.

c) Solve the problem by method of characteristic $pz + q = 1$ with initial data $y = x, z = \frac{x}{2}$.

d) Prove that the general solution of the linear partial differential equation $Pp + Qq = R$ is $F(u, v) = 0$ where F is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ form a solution of the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

e) Using convolution theorem prove that

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m, n > 0.$$

Hence deduce that

$$\int_0^{\pi/2} \sin^{2m-1} \theta \cdot \cos^{2n-1} \theta d\theta = \frac{1}{2} B(m, n).$$

f) Prove that

$$L^{-1} \left\{ \frac{s}{s^4 + s^2 + 1} \right\} = \frac{2}{\sqrt{3}} \sinh \frac{t}{2} \cdot \sin \frac{\sqrt{3}}{2} t.$$

GROUP-C

[Marks : 20]

3. Answer any two questions: $10 \times 2 = 20$

a) i) Discuss compatibility of two first order partial differential equations. Derive a necessary and sufficient condition for the compatibility of the two first order partial differential equations $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$.

ii) Show that the equations $xp - yq = x$ and $x^2p + q = xz$ are compatible and find their solution. $(1+5)+4$

b) i) Solve the heat conduction problem for a finite rod of length l , $\frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0$, $0 \leq x \leq l$, $t \geq 0$ subject to the boundary conditions $T(0, t) = T(l, t) = 0$, $t \geq 0$ and the initial condition $T(x, 0) = f(x)$, $0 \leq x \leq l$.

ii) Prove that the solution is unique.

iii) Solve $T_t = T_{xx}$, $0 < x < l$, $t > 0$, with $T(0, t) = T(l, t) = 0$ and $T(x, 0) = x(l-x)$. $4+3+3$

c) i) Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

ii) Define integral strip and characteristic strip. Prove that along every characteristic strip of the equation $f(x, y, z, p, q) = 0$, the function $f(x, y, z, p, q)$ is constant. $5+5$

d) Find $L\{\operatorname{erf} \sqrt{t}\}$ and hence show that

$$L\{t \cdot \operatorname{erf}(2\sqrt{t})\} = \frac{3s+8}{s^2(s+4)^{3/2}}. \quad 5+5$$

- e) Solve the differential equation $y''(t) + 2y'(t) + y(t) = 3te^{-t}$ subject to $y(0) = 4, y'(0) = 2$ using Laplace transforms.

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