

U.G. 5th Semester Examination - 2024

MATHEMATICS

[HONOURS]

Discipline Specific Elective (DSE)

Course Code : MATH-H-DSE-T-1A

(Group Theory-II)

[New Syllabus]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

GROUP-A

(Marks : 20)

1. Answer any ten questions: $2 \times 10 = 20$
- a) If G be an abelian group and $f : G \rightarrow G$ be such that $f(x) = x^{-1}$ then show that f is an automorphism.
 - b) Define characteristic subgroup of a group.
 - c) Prove that center of a group G is a characteristic subgroup of G .
 - d) Define external direct product of groups.

[Turn over]

- e) Is the direct product $\mathbb{Z} \times \mathbb{Z}$ a cyclic group? Justify your answer.
- f) Show that all abelian groups of order 22 are cyclic.
- g) Prove that $\mathbb{Z}$ under addition is not isomorphic to $\mathbb{Q}$ under addition.
- h) Find the number of inner automorphism of the group S_4 .
- i) Find the class equation of a noncommutative group of order 8.
- j) If G be a finite group having only two conjugacy classes, then show that $|G| = 2$.
- k) State Sylow's First Theorem.
- l) Define p -group with an example.
- m) Let G be a noncommutative group of order p^3 , p a prime. Prove that $|Z(G)| = p$.
- n) Let G be a group of order 99. Prove that G has a unique normal subgroup H of order 11.
- o) Show that center of a group of order 65 is isomorphic to $\mathbb{Z}_{65}$.

GROUP-B

(Marks : 20)

2. Answer any four questions: 5×4=20
- a) Show that a characteristic subgroup of a group G is a normal subgroup of G . Is the converse true? Justify your answer. 3+2
 - b) Let $G_1, G_2, \dots, G_n$ be n groups. Prove that the group $G = G_1 \times G_2 \times \dots \times G_n$ is abelian if and only if each of the groups G_1, G_2, G_n is abelian. 5
 - c) Let S be a G -set, where G is a group and S is a nonempty set. Define a relation ρ on S by for all $a, b \in S$, $a \rho b$ if and only if $ga = b$ for some $g \in G$. Prove that ρ is an equivalence relation on S . 5
 - d) Show that every group of order 15 is commutative. 5
 - e) Find all abelian groups of order 96. 5
 - f) If G is a group of order pn , where p is prime such that $p \geq n$, then show that G has a normal subgroup of order p . 5
 - g) Prove that no group of order 30 is simple. 5

GROUP-C

(Marks : 20)

3. Answer any two questions: $10 \times 2 = 20$

a) i) Let G be a group and Z be the centre of the group G . Prove that set of all inner automorphisms of G is isomorphic to the quotient group G/Z . 5

ii) If $|Aut(G)| > 1$, then show that $|G| > 2$. 3

iii) Let G be a group and $|Inn(G)| = 1$. Is G abelian? Justify your answer. 2

b) i) Prove that for any group G , $\left| \frac{G}{Z(G)} \right| \neq 91$. 5

ii) Show that upto isomorphism there are only three non-isomorphic abelian group of order 8. 5

c) i) If G be a group of order pq , where p, q are primes such that $p > q$ and q does not divide $p-1$, then show that G is a cyclic group. 5

ii) Find the number of elements of order 5 in the group $\mathbb{Z}_{25} \times \mathbb{Z}_5$. 5