

U.G. 6th Semester Examination - 2025

MATHEMATICS

[HONOURS]

Discipline Specific Elective (DSE)

Course Code : MATH-H-DSE-T-04A

(Point Set Theory)

[New Syllabus]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**Candidates are required to give their answers in their own words as far as practicable.**The symbols and notations have their usual meanings.*

1. Answer any ten questions: $2 \times 10 = 20$
- a) Let X be an infinite set. Then show that
$$\tau = \{A \subseteq X : A^c = X - A \text{ is infinite}\}$$
is not a topology on the set X .
- b) Show that every regular space may not be T_2 .
- c) Find a base for the discrete topology and a base for the indiscrete topology on a nonempty set X .
- d) Give an example to show that
$$\text{int}(A \cup B) \neq \text{int}(A) \cup \text{int}(B),$$
where A, B are subsets of a topological space X .
- e) If U and V be two disjoint open subsets of a topological space X , then show that
$$\text{cl}(U) \cap V = \emptyset \text{ and } \text{cl}(V) \cap U = \emptyset.$$

[Turn Over]

- f) Show that every Co-finite infinite topological space is not T_2 .
- g) Let (Y, τ_Y) be a subspace of a topological space (X, τ) . If β_X is a base for the topology τ of X , then show that

$$\beta_Y = \{A \cap Y : A \in \beta_X\}$$

is a base for the topology τ_Y of Y .

- h) Let (X, τ) and (Y, δ) be two topological spaces. Then show that

$$\beta = \{U \times V : U \in \tau, V \in \delta\}$$

is a base for some topology on $X \times Y$.

- i) Show that every second countable space is separable.
- j) Show that a connected space has no proper nonempty open subset which is also closed.
- k) Give an example of a first countable space which is not second countable.
- l) If x and y are two distinct points of a T_0 space X , then show that $cl(\{x\}) \neq cl(\{y\})$.
- m) Show that a compact subset of a compact T_2 space is closed.
- n) Give an example to show that the union of two disconnected subsets of a Space X may be connected.
- o) Show that every regular T_1 space is T_2 .

2. Answer any four questions:

5×4=20

- a) Let β be a cover for a nonempty set X , then show that the following conditions are equivalent:
- β is a base for some topology on X .
 - for $B_1, B_2 \in \beta$ and $x \in B_1 \cap B_2$, there exists a $B \in \beta$ such that $x \in B \subseteq B_1 \cap B_2$. 5
- b) For a subset A of a topological space X , establish the relations:
- $cl(X - A) = X - int(A)$, 2+3
 - $cl(A) = A \cup A'$.
- c) i) If X is a T_2 space, then show that the continuity of the mapping $f: X \rightarrow X$ implies $B = \{x \in X : f(x) \neq x\}$ is an open subset of X .
- ii) If Y be a subspace of a topological space X , then show that $cl_Y(A) = cl_X(A) \cap Y$, where $A \subseteq Y$. 3+2
- d) Prove that a topological space X is regular if and only if for each open subset U of X and for each point $x \in U$, there exists an open set V of X such that $x \in V \subseteq cl(V) \subseteq U$. 5
- e) Show that the product space $X = X_1 \times X_2$ is compact if and only if X_1 and X_2 are compact spaces. 5
- f) Define the projection maps and show that they are continuous, open and onto. 5

3. Answer any two questions:

10×2=20

- a) i) Let (X, d) be an infinite metric space. Then show that the set β of all open balls (open spheres) of X forms a base for some topology τ on X that is also T_2 .
- ii) Show that every compact T_2 space is regular. 5+5

- b) i) If $\{\tau_\alpha : \alpha \in \Delta\}$ is an arbitrary family of topologies on X , then show that

$$\bigcap_{\alpha \in \Delta} \tau_\alpha = \left\{ \bigcap_{\alpha \in \Delta} U_\alpha : U_\alpha \in \tau_\alpha \text{ for each } \alpha \in \Delta \right\}$$

is a topology on X . Give an example to show that the union of two topologies on X may not be a topology on X ?

- ii) Let A be a connected subset and B be any subset of a topological space X such that

$$A \subseteq B \subseteq \bar{A}.$$

Show that B is connected. 5+5

- c) i) Show that a topological space X is T_2 if and only if every convergent net in X converges to a unique point of X .
- ii) Show that every closed subset of a compact space is compact. 7+3