

U.G. 6th Semester Examination - 2025

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-13

(Metric Spaces and Complex Analysis)

[Old Syllabus]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**The symbols and notations have their usual meanings.*1. Answer any ten questions: $2 \times 10 = 20$

- a) Show that $f(z) = \frac{1}{z^2}$ is uniformly continuous in $\frac{1}{2} \leq |z| \leq 1$.
- b) Show that the union of a finite number of closed sets in a metric space is closed.
- c) Let $f = u + iv$ be analytic in a domain D . Show that f is constant in D if $\text{Im } f$ is constant in D .
- d) Show that a closed subset of a compact metric space is compact.
- e) Verify whether $w = z^{\frac{1}{3}}$ is algebraic function or transcendental function.

[Turn Over]

f) Given that G is an open subset of a metric space (X, d) . Show that G is a union of open balls in (X, d) .

g) Show that the set of all positive integers constitutes an incomplete metric space if

$$\rho(m, n) = \left| \frac{1}{m} - \frac{1}{n} \right|.$$

h) Show that the function $f(z) = \frac{\bar{z}}{z}$ does not have a limit as $z \rightarrow 0$.

i) Show that the set $X = \mathbb{R}$ with the metric

$$\rho(x, y) = \frac{|x - y|}{1 + |x - y|} \text{ is bounded.}$$

j) If z is a complex number such that $z\bar{z} = 1$ then find the value of $|1 + z|^2 + |1 - z|^2$.

k) Let (x, d) be a metric space and $A, B \subset X$. Show that $d(A \cup B) \leq d(A) + d(A, B) + d(B)$.

l) Evaluate $\oint_{|z|=2} \frac{z dz}{(9 - z^2)(z + i)}$.

m) Verify whether $\frac{e^{\frac{4\pi i}{5}} - 1}{e^{\frac{4\pi i}{5}} + 1}$ is a root of the equation

$$(1 + z)^5 = (1 - z)^5.$$

n) Prove or disprove: Continuous image of a locally connected metric space is locally connected.

o) Show by an example that the union of an infinite number of closed sets need not be closed.

2. Answer any four questions: 5 × 4 = 20

a) A sequence $\{x_n\}$ in a metric space x is convergent and converges to x . Prove that $\{x\} \cup \{x_n : n = 1, 2\}$ is a compact subset of x .

b) Let $f(z) = \sqrt{xy}$. Show that though C-R equations are satisfied at origin but $f'(0)$ does not exist.

c) Evaluate $\int_{-2+i}^{5+3i} z^3 dz$.

d) State and prove Cantor's intersection theorem.

e) If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $f(z) = u + iv$ is analytic.

f) Show that a compact metric space is always second countable.

3. Answer any two questions: 10 × 2 = 20

a) i) Prove that a sequence $x_n \rightarrow x$ in the metric space $C[a, b]$ if and only if the corresponding sequence of real valued functions $\{x_n(t)\}$ converges uniformly to $x(t)$ on $[a, b]$.

ii) Let $f(z) = e^{-z^{-1}}$, $z \neq 0$
 $= 0$, $z = 0$

Show that though Cauchy-Riemann equations are satisfied at $(0, 0)$, $f'(0)$ does not exist. 5+5

b) i) Show that $\left\{ \left(1 + \frac{1}{n} \right)^n \right\}$ is a convergent sequence in real number space with usual

metric, and hence obtain $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{2n}$.

ii) Determine the region of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n} \quad 5+5$$

c) i) Show that boundedness and total boundedness are equivalent in $\mathbb{R}^n$.

ii) Define uniform continuity of a function $f(z)$. Show that the continuity of $f(z)$ on a set S implies uniform continuity of $f(z)$ on S if S is bounded and closed in $\mathbb{C}$.

5+5