

U.G. 5th Semester Examination - 2024

MATHEMATICS

[PROGRAMME]

Discipline Specific Elective (DSE)

Course Code : MATH-G-DSE-T-1(A&B)

[New Syllabus]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours.

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols and notations have their usual meanings.

Answer all the questions from Selected Option.

OPTION-A

MATH-G-DSE-T-1A

(Group Theory and Linear Algebra)

1. Answer any ten questions: $2 \times 10 = 20$
 - a) Let G be a cyclic group of order 30 generated by a . Find the order of the cyclic subgroup generated by a^{18} .
 - b) Prove that every group of prime order is cyclic.
 - c) Find the index of the subgroup $H = \{e, a^2\}$ in the group $G = \{e, a, a^2, a^3, a^4, a^5\}$, where G is cyclic.

[Turn over]

- d) Verify whether $\{0, 2, 4\}$ is a subgroup of $\mathbb{Z}_6$ under addition modulo 6.
- e) What is the dual basis of $\{(1,1), (1,-1)\}$ in $\mathbb{R}^2$?
- f) Verify whether the set $\{1+x, 1-x, x^2\}$ is linearly independent in the vector space of polynomials of degree at most 2.
- g) If $G = \mathbb{Z}_{12}$, find the order of the element 5 mod 12.
- h) Find the number of left coset of the subgroup $\{e, (12)\}$ in S_3 .
- i) Find all normal subgroups of the group $\mathbb{Z}_4$.
- j) Define the characteristic polynomial of a 2×2 matrix.
- k) Let $(G, *)$ be a group and $a, b \in G$. If $a^2 = e$ and $a * b^2 * a = b^3$, prove that $b^5 = e$.
- l) Define basis of a vector space with an example.
- m) Show that matrices A and A^T have the same eigenvalues.
- n) Verify whether the mapping $T(x, y) = (x^2, y^2)$ is a linear transformation.
- o) Find $k \in \mathbb{R}$ so that the set $S = \{(1, 2, 1), (k, 3, 1), (2, k, 0)\}$ is linearly dependent in $\mathbb{R}^3$.

2. Answer any four questions:

5×4=20

- a) Let M and N be normal subgroups of a group G such that $M \cap N = \{e\}$. Prove that $mn = nm$ for all $m \in M$ and all $n \in N$.
- b) Prove that the subset $H = \{i, (1234), (1432), (13)(24)\}$ of S_4 forms a cyclic subgroup of the symmetric group S_4 . Find the generators of H .
- c) Consider the bases $S = \{(1,2), (2,3)\}$ and $S' = \{(1,3), (1,4)\}$ of $\mathbb{R}^2$. Find the change-of-basis matrix P from S to S' .
- d) Find the matrix representation of $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$T(x, y, z) = (2x - 4y + 9z, 5x + 3y - 2z)$$
relative to the usual basis.
- e) Show that Subgroup of a Cyclic group is Cyclic.
- f) Find a matrix P such that $P^{-1}AP$ is a diagonal matrix, where

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

- g) State and prove the Lagrange's theorem.
- h) Determine the linear mapping $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which maps the basis vectors $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ of $\mathbb{R}^3$ to $(1, 1, 1)$, $(1, 1, 1)$, $(1, 1, 1)$ respectively. Verify that $\dim \ker T + \dim \text{Im } T = 3$.

3. Answer any two questions: $10 \times 2 = 20$

- a) i) Find all elements of order 10 in the group $(\mathbb{Z}_{30}, +)$.
- ii) Prove that a group of order 27 must have a subgroup of order 3.
- b) i) Find the dimension of the subspace S of $\mathbb{R}^3$ defined by
 $S = \{(x, y, z) \in \mathbb{R}^3 : x + 2y = z, 2x + 3z = y\}$.
- ii) Show that the set of vectors $S = \{(1, 2, 0), (2, 1, 3), (1, 1, 1), (2, 3, 1)\}$ is linearly dependent in $\mathbb{R}^3$. Find a linearly independent subset S_1 of S such that $L(S_1) = L(S)$.
- c) i) Let G be a group in which $(ab)^3 = a^3b^3$ for all $a, b \in G$. Prove that $H = \{x^3 : x \in G\}$ is a normal subgroup of G .
- ii) If an abelian group G of order 10 contains an element of order 5, prove that G must be a cyclic group.
- d) i) Prove the eigenvalues of a real symmetric matrix are all real.
- ii) A linear mapping $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is defined by
 $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3)$,
 $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find the matrix of T relative to the ordered bases, $((0, 1, 1), (1, 0, 1), (1, 1, 0))$ of $\mathbb{R}^3$ and $((1, 0), (0, 1))$ of $\mathbb{R}^2$.

OPTION-B

MATH-G-DSE-T-1B

(Complex Analysis)

1. Answer any ten questions: $2 \times 10 = 20$

- a) By the definition of limit show that
 $\lim_{z \rightarrow 1} |z - 2| = 1$.
- b) Show that if $f(z) = \frac{1}{z}$ is not bounded near $z = 0$.
- c) If $z = \sin \alpha \cos \theta + i \sin \alpha \sin \theta$, then find the greatest value of $|z|$.
- d) If $z = \cos(\sin \frac{\pi}{2}) + i \sin(\sin \frac{\pi}{2})$ then find the least value of $|\operatorname{Im} z|$.
- e) Find the equation of a straight line in complex plane non-parallel to the axes.
- f) Find $\frac{df}{dz}$ from definition if $f(z) = z^2$.
- g) Discuss the continuity of the function
 $f(z) = \frac{\sin z}{z}$, if $z \neq 0$ and $f(z) = 1$ if $z = 0$
 at the point $z = 0$.

- h) Find the radius of convergence of $1 + \alpha z + \alpha^2 z^2 + \alpha^3 z^3 + \dots$.
- i) Evaluate $\int_{|z|=1} |z-1| |dz|$.
- j) Show that $\int_{\gamma} z dz = 0$, where $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$.
- k) If fg is analytic at a point $z = z_0$, can we conclude that each of the functions f and g is also analytic at z_0 ? Justify your answer.
- l) Check whether $f(z) = x^2 + iy^2$ is analytic at the origin.
- m) Evaluate $\int_{\gamma} \frac{1}{z-1} dz$, where $\gamma = \{z: |z-1| = 1\}$.
- n) State the Cauchy-Reimann Equations for analyticity of a function $f(z)$.
- o) Is $f(z) = \cos z$ a bounded function? Justify your answer.
2. Answer any four questions: $5 \times 4 = 20$
- a) Show that $f(z) = \bar{z}$ is nowhere differentiable and $g(z) = p(z)$ is everywhere differentiable, where $p(z)$ is any polynomial in z .

- b) State Cauchy's integral formula. Use this formula to show that $\int_{|z|=1} \frac{\sin z}{z} dz = 2\pi i \sin 0 = 0$.
- c) If $f = u + iv$ be analytic, find v , when $u(x, y) = x + y$.
- d) State and prove Liouville's theorem.
- e) Evaluate the following line integral: $\int_{\gamma} x dz$, where γ is the straight line segment from 0 to $a + ib, a, b \in \mathbb{R}$.
- f) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} (2^n + 3^n) z^n$.
3. Answer any two questions: $10 \times 2 = 20$
- a) i) Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be defined and continuous in some neighbourhood of a point $z = z_0$. Then show that both $u(x, y)$ and $v(x, y)$ are continuous at z_0 .
- ii) Find an upper bound of $\left| \int_{\gamma} e^z dz \right|$ where $\gamma(t) = t^2 + 2ti$, for $0 \leq t \leq 2$.

5+5

- b) i) State and prove the fundamental theorem of algebra.
- ii) State Cauchy's integral formula. Can we apply this formula to evaluate $\int_{\gamma} \frac{\sin \frac{1}{z}}{z} dz$, where $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$? Give reason in support of your answer. 5+5
- c) i) Check the differentiability of $f(x + iy) = x^2 + y^2 + ixy$, using the C-R equations. What can be said about the analyticity of f ?
- ii) Verify the differentiability of $f(z) = z \operatorname{Re} z$ at $z = 0$. 6+4
- d) i) Let $f(z) = \sum a_n (z - z_0)^n$ be a series with radius of convergence R . If $r < R$ and $M(r) = \max\{|f(z)| : |z - z_0| = r\}$, then show that for each
- $$k \in \mathbb{N}, |a_k| \leq \frac{M(r)}{r^k}, k = 1, 2, 3, \dots$$
- ii) Expand in Taylor series $f(z) = e^{z-1}$ about $z = 0$. 6+4
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