

U.G. 4th Semester Examination - 2024

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-9

(Multivariate Calculus and Tensor Analysis)

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**The symbols and notations have their usual meanings.*

1. Answer any ten questions:

 $2 \times 10 = 20$

a). Show that for the function $f(x, y) = |x| + |y|$, partial derivatives f_x and f_y do not exist at $(0, 0)$ but $f(x, y)$ is continuous at $(0, 0)$.

b). Prove that δ_j^i is a tensor of type $(1, 1)$.

c). If u and v are function of x, y, z satisfying the relationships

$$uv = 3x - 2y + z, v^2 = x^2 + y^2 + z^2, \text{ show that}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0.$$

d). Evaluate the integral

$$\iiint \cos(x + 2y + 3z) \, dx \, dy \, dz \text{ over the region}$$

$$R: \{0 \leq x \leq 2, 1 \leq y \leq 3, |z| \leq 1\}.$$

[Turn Over]

e) If a_s^r in a double system such that $a_m^r a_s^m = \delta_s^r$, show that $|a_s^r| = \pm 1$.

f) Prove that the function

$$f(x, y) = \begin{cases} \frac{x^3 + y^3}{x - y} & \text{when } x \neq y \\ 0 & \text{when } x = y \end{cases} \text{ is not continuous}$$

at $(0, 0)$.

g) Write down the quotient law for contravariant and covariant tensors of second order.

h) Express $\int_0^{\frac{\pi}{2}} dx \int_0^{\cos x} x^2 dy$ as a double integral and evaluate it.

i) State Young's theorem for the equality of mixed partial derivatives.

j) If $\vec{\nabla} f(x, y, z) = 2xyze^{x^2} \hat{i} + ze^{x^2} \hat{j} + ye^{x^2} \hat{k}$ and $f(0, 0, 0) = 7$, evaluate $f(1, 1, 2)$.

k) Evaluate $\oint_{\Gamma} (e^x dx + 2y dy - dz)$ by using Stokes's theorem, Where Γ represents the curve $x^2 + y^2 = 4, z = 2$.

l) If the equality $a_j^i v_i = \sigma v_j$ holds for every covariant vector v_j , where σ is a scalar, show that $a_j^i = \sigma \delta_j^i$.

m) If $f(x, y) = k$ and $S = [a, b] \times [c, d]$, where k is a constant, then find the value of $\iint_S f(x, y) dx dy$ over S .

n) If A^i and B_i are the components of contravariant and covariant vectors respectively, then the sum $A^i B_i$ is an invariant.

o) Give an example of a continuous function $f(x, y)$ which does not have partial derivatives of the first order.

2. Answer any four questions:

5 × 4 = 20

a) If $f(x, y) = \begin{cases} 1 & \text{when } xy \neq 0 \\ 0 & \text{when } xy = 0 \end{cases}$

Show that,

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y)$$

but $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist.

b) Find the directional derivative of $f(x, y, z) = x^2 + 2y^2 - 3z^2$ at $(1, 1, 1)$ along the line joining the points $(0, 0, 0)$ and $(1, 1, 1)$.

c) If $a_{pqr} x^p x^q x^r = 0$ for all values of independent variables $x^1, x^2, \dots$ and a_{pqr} are constants, show that

$$a_{kjl} + a_{jki} + a_{ikj} + a_{ljk} + a_{klj} + a_{jlk} = 0.$$

d) Evaluate the double integral $\iint_D e^{x^2} dx dy$, where the region $D = \{(x, y) \in R^2: 2y \leq x \leq 2 \text{ and } 0 \leq y \leq 1\}$.

e) Evaluate $I = \int_0^1 dx \int_0^x \sqrt{x^2 + y^2} dy$ by transforming to polar co-ordinates.

f) Prove that the following three functions u, v, w are functionally related and find the relation connecting them,

$$\text{when } u = \frac{x}{y-z}, v = \frac{y}{z-x}, w = \frac{z}{x-y}.$$

g) Let $\frac{4}{x} + \frac{9}{y} + \frac{16}{z} = 25$. Use Lagrange's method to find values of x, y, z such that $x + y + z$ is minimum.

3. Answer any two questions: 10×2=20

a) i) If $g = |g_{ij}| > 0$, then Show that

$$\left\{ \frac{i}{l} \right\} = \frac{\partial \log(\sqrt{g})}{\partial x^l}.$$

ii) If $f(x, y) = \begin{cases} xy & \text{when } |x| \geq |y| \\ -xy & \text{when } |x| < |y| \end{cases}$

Show that, $f_{yx}(0,0) \neq f_{xy}(0,0)$. Which conditions of Schwarz's theorem are not satisfied by f . 5+5

b) i) A function $f(x, y)$ having continuous second order partial derivatives when expressed in terms of new variables u and v defined by $x = \frac{1}{2(u+v)}$ and $y^2 = uv$ becomes $g(u, v)$. Prove that,

$$\frac{\partial^2 g}{\partial u \partial v} = \frac{1}{4} \left(\frac{\partial^2 f}{\partial x^2} + \frac{2x}{y} \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} + \frac{1}{y} \frac{\partial f}{\partial y} \right).$$

ii) Show that the function,

$$f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & \text{when } (x, y) \neq (0, 0) \\ 0 & \text{when } x^2 + y^2 = 0 \end{cases}$$

although continuous at $(0, 0)$ is not differentiable there. 5+5

c) i) If $H(x, y)$ be a homogeneous function of x and y of degree n having continuous partial derivatives and

$$u(x, y) = (x^2 + y^2)^{-\frac{n}{2}}. \text{ Show that } \frac{\partial}{\partial x} \left(H \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(H \frac{\partial u}{\partial y} \right) = 0.$$

ii) If a vector has components $\frac{dx}{dt}, \frac{dy}{dt}$ in rectangular cartesian coordinates (x, y) . Show that $\frac{dr}{dt}, \frac{d\theta}{dt}$ are its components in polar coordinates (r, θ) . 5+5