

U.G. 5th Semester Examination - 2024

MATHEMATICS

[PROGRAMME]

Discipline Specific Elective (DSE)

Course Code : MATH-G-DSE-T-1(A&B)

[Old Syllabus]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols and notations have their usual meanings.

Answer all the questions from Selected Option.

OPTION-A

MATH-G-DSE-T-1A

(Matrices and Linear Algebra)

1. Answer any ten questions: $2 \times 10 = 20$
 - i) Show that the eigenvalues of a real symmetric matrix are purely imaginary or zero.
 - ii) A transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $T(x, y) = (x, 0)$. Check whether it is linear or not.
 - iii) If W_1, W_2, W_3 are three subspaces of a vector space V over F , then find the smallest subspace contained in each of the above subspaces.

[Turn Over]

- iv) If c is a scalar, prove that $(cA)^t = cA^t$.
- v) If U and W are two subspaces of a vector space V over F then check whether $U \cup W$ is a subspace of V or not.
- vi) Check whether $\{(x, x+1) : x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$ or not.
- vii) Is the system of equations $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ solvable? Justify your answer.
- viii) Consider the vectors $\alpha = (4, 3, 5)$, $\beta = (0, 1, 3)$, $\gamma = (2, 1, 1)$. Examine if α is a linear combination of β and γ .
- ix) If $A = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}$, then show that A^2 is symmetric.
- x) Express the matrix $\begin{pmatrix} 4 & 5 & 1 \\ 3 & 7 & 2 \\ 1 & 6 & 8 \end{pmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.
- xi) If A and B are two square matrices of the same order, then justify whether $(A+B)^2 = A^2 + 2AB + B^2$ is true.

- xii) If $\alpha = (a, b)$, $\beta = (a+b, b-a)$ and $\gamma = (b, a)$, find suitable scalars p and q such that $p\alpha + q\beta = \gamma$ in $\mathbb{R}^2$.
- xiii) If the vectors $(0, 1, a)$, $(1, a, 1)$, $(a, 1, 0)$ of the vector space $\mathbb{R}^3$ be linearly dependent, then find the value of a .
- xiv) Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$.
- xv) If x be an eigenvalue of an orthogonal matrix A , then show that $\frac{1}{x}$ is also an eigenvalue of A .

2. Answer any four questions:

$5 \times 4 = 20$

- i) Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

- ii) Verify whether the system of equations

$$2x - 3y + z = 1$$

$$x + 2y - 3z = 4$$

$$4x - y - 2z = 8$$

is consistent. If so, solve it.

- iii) Let V be a vector space over a field F and let $\alpha \in V$. Show that the set $W = \{C\alpha \mid C \in F\}$ forms a subspace of V .

iv) Find the row rank of the matrix:

$$A = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 6 & 9 \\ 1 & 1 & 2 & 6 \end{pmatrix}.$$

v) If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, show that $A^2 - 4A - 5I_3 = O$.

Hence obtain a matrix B such that $AB = I_3$.

vi) Show that the vectors $\{(1, 2, 1), (2, 1, 0), (1, -1, 2)\}$ form a basis of the vector space $\mathbb{R}^3$ over $\mathbb{R}$.

3. Answer any two questions: $10 \times 2 = 20$

i) a) Solve the following system of equations by matrix method:

$$2x - y + 3z = 8$$

$$-x + 2y + z = 4$$

$$3x + y - 4z = 0.$$

b) Find a basis of $\mathbb{R}^4$ containing the vectors $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$ and express the vector $(1, 2, 3, 4)$ as a linear combination of the basis vectors. $5+5$

ii) a) Prove that, there cannot exist real 3×3 matrices A and B such that

$$AB - BA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

b) Solve the following system of equations by Cramer's rule:

$$x + 2y - 3z = 1$$

$$2x - y + z = 4$$

$$x + 3y = 5 \quad 5+5$$

iii) a) Determine the linear transformation T on $\mathbb{R}^3$ which maps $(0, 1, 1)$ to $(1, 0, 1)$; $(1, 0, 1)$ to $(2, 3, 4)$ and $(1, 1, 0)$ to $(1, 2, 3)$.

b) Using Cayley-Hamilton theorem find the inverse of the matrix $A = \begin{bmatrix} 7 & 4 \\ 4 & 7 \end{bmatrix}$. $5+5$

iv) a) Show that the planes passing through the origin is a proper subspace of $\mathbb{R}^3$.

b) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z).$$

Find $\text{rank}(T)$ and $\text{nullity}(T)$. $5+5$

OPTION-B
MATH-G-DSE-T-1B
(Complex Analysis)

1. Answer any ten questions: 2×10=20

- a) Show that $\lim_{z \rightarrow 2i} \frac{z^2 + 4}{z - 2i} = 4i$.
- b) $\int_C \frac{dz}{z}$, where C is the contour $z(t) = e^{it}$, $t \in [0, 2\pi]$.
- c) Show that the function $f(z) = xy + iy$ is continuous everywhere but not differentiable.
- d) Show that $\exp(\log z) \neq \log(\exp z)$.
- e) Show that an analytic function with constant modulus is constant.
- f) Check differentiability of $f(z) = |z|$ at $z = 0$.
- g) If $|z_1| = |z_2| = 1$ and $\text{amp}(z_1) + \text{amp}(z_2) = 0$, then show that $z_1 z_2 = 1$.
- h) Show that the function $f(z) = z^3$ is analytic in a domain D of the complex plane C .
- i) From definition prove that $\sin z$ is unbounded.
- j) Find values of i^i .
- k) Show that the function $f(z) = x + y$ is not analytic at any point.
- l) Draw the graph of the contour: $z(t) = t + it$, $t \in [0, 1]$.
- m) If $a = \cos \theta + i \sin \theta$, obtain the value of θ in $[0, \pi]$ such that $a^3 = i$.

2. Answer any four questions: 5×4=20

- a) State and prove Liouville's theorem. Hence show that, every polynomial of degree n has n zeros in the complex plane.
- b) State and prove Cauchy integral formula for a function of a complex variable.
- c) Evaluate $\int_C \bar{z} dz$, where C is the upper half of the circle $|z| = 1$, $z = -1$ to $z = 1$.
- d) Let $u(x, y) = e^x \cos y$. Determine a function $v(x, y)$ such that $f = u + iv$ is analytic.
- e) Let $f(z) = |z^2|$. Show that the derivative of $f(z)$ exists only at the origin.
- f) Let $f(z) = \frac{1}{z^2}$ and Γ be the straight line joining the points i and $2 + i$. Show that $\left| \int_{\Gamma} f(z) dz \right| \leq 2$.

3. Answer any two questions: 10×2=20

- a) i) For what values of z does the series

$$\sum_{n=0}^{\infty} (-1)^n (z^n + z^{n+1})$$

converge and find its sum.

- ii) Draw diagram and evaluate: 5+5

$$\int_C (z-1) dz, \text{ where } C: z(t) = \begin{cases} t+it, & t \in [0, 2] \\ 2+(4-t), & t \in [2, 3] \\ (5-t)+i, & t \in [3, 4] \end{cases}$$

- b) i) Let f be analytic in a simple connected region R and let α, β be any two points in R . Then show that $\int_{\alpha}^{\beta} f(z) dz$ is independent of the path joining α and β .
- ii) Show that for even complex number z , $|e^z| \leq e^{|z|}$. 6+4

- c) i) For what values of z , do the function w defined by the following equation cease to be analytic?

$$z = -e^{-v} (\cos u + i \sin u), \quad W = u + iv \text{ and } z = \sin u \cos(iv) + \cos u \sin(iv).$$

- ii) Evaluate $\int_C \frac{dz}{(z-\alpha)^m}$, where m is an integer and $C: |z-\alpha|=r$, described in the positive sense. 6+4

- d) i) Evaluate $\int_{|z|=1} \frac{dz}{z+2}$ and deduce that

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0.$$

- ii) Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{(z^2+1)^n}. \quad \text{5+5}$$