

U.G. 4th Semester Examination - 2025**MATHEMATICS****[MAJOR]****Course Code : MATH-MJ-T-5****(Algebra-II)****[NEP-2020]**

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

*The figures in the right-hand margin indicate marks.
The notations and symbols have their usual meanings.*

GROUP-A

1. Answer any ten questions: $2 \times 10 = 20$
- a) Give an example of an infinite group in which every element has finite order.
 - b) Prove that every proper subgroup of a group of order 6 is cyclic.
 - c) Prove that center of a group G is a characteristic subgroup of G .
 - d) Is the external direct product $\mathbb{Z}_6 \times \mathbb{Z}_4$ cyclic? Justify your answer.
 - e) Define internal direct product of groups.
 - f) Show that center of a group of order 65 is isomorphic to $\mathbb{Z}_{65}$.
 - g) State Sylow's First Theorem.
 - h) Show that the mapping $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3^x$, $x \in \mathbb{R}$ is not a ring homomorphism.
 - i) Show that $\frac{\mathbb{Z}}{3\mathbb{Z}} \cong \frac{4\mathbb{Z}}{12\mathbb{Z}}$.

[Turn over]

- j) Determine k so that $(1,3,1)$, $(2, k, 0)$ and $(0,4,1)$ are linearly dependent.
- k) Is the mapping $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T((x, y)) = (x^2, y^2) \forall (x, y) \in \mathbb{R}^2$ a linear transformation? Justify your answer.
- l) Let V be a vector space with a basis $\{e^{3t}, te^{3t}, t^2 e^{3t}\}$ and $D: V \rightarrow V$ be defined by $Df(t) = \frac{d}{dt} f(t)$, $f(t) \in V$. Find the matrix of D in the given basis.
- m) Is the matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ diagonalisable? Justify your answer.
- n) Use Cayley Hamilton theorem to compute A^{-1} , where $A = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$.
- o) If $F(x) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$, find $\text{adj } F(x)$ as a matrix polynomial.

GROUP-B

2. Answer any **four** questions: $5 \times 4 = 20$
- a) Let G and G' be two groups and let $\phi: G \rightarrow G'$ be an onto homomorphism. Prove that $G / \text{Ker } \phi \cong G'$.
- b) Prove that for any group G , $|G / Z(G)| \neq 91$.

- c) Examine whether the ring of matrices $\left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ is a field.
- d) Show that the set of vectors $S = \{(1,2,0), (3,-1,1), (4,1,1)\}$ is linearly dependent in $\mathbb{R}^3$. Apply Deletion theorem to find a proper subset S_1 of S such that $L(S_1) = L(S)$.
- e) Find a basis for the row space of the matrix $\begin{pmatrix} 2 & 3 & 1 & 0 \\ 4 & 0 & 6 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.
- f) Let V and W be vector spaces over a field F with V finite dimensional. If $T: V \rightarrow W$ is a linear mapping, then prove that $\dim \text{Ker } T + \dim \text{Im } T = \dim V$.

GROUP-C

3. Answer any **two** questions: $10 \times 2 = 20$
- a) i) For a group G , prove that $\text{Inn}(G)$ is a normal subgroup of $\text{Aut}(G)$. 3
- ii) Examine whether the ring of matrices $\left\{ \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ contains the unity. 2
- iii) Extend the set $\{(1, 1, 1, 1), (1, -1, 1, -1)\}$ to a basis of $\mathbb{R}^4$. 5

- b) i) If G be a finite abelian group and d be a positive divisor of $o(G)$, then prove that G has a subgroup of order d . 5
- ii) Show that every subgroup of a cyclic group is a characteristic subgroup. 5
- c) i) Prove that intersection of two ideals of a ring R is an ideal of R . 2
- ii) Show that the mapping $f: \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{Z}[\sqrt{3}]$ defined by $f(a+b\sqrt{2}) = a+b\sqrt{2}$ is a group homomorphism but not a ring homomorphism. 3
- iii) Use Gram-Schmidt process to obtain an orthonormal basis of the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product, generated by the linearly independent set $\{(1,1,0,1), (1,1,0,0), (0,1,0,1)\}$. 5
- d) i) Diagonalise the matrix $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$. 5
- ii) If a group G be the internal direct product of its subgroups H and K , then prove that $G \cong H \times K$. 5