

U.G. 3rd Semester Examination - 2024**MATHEMATICS****[HONOURS]****Course Code : MATH-H-CC-T-05****(Theory of Real & Vector Functions)****[CBCS]**

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**Symbols and notations have their usual meanings.*

1. Answer any ten questions:

 $2 \times 10 = 20$

- i) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on $\mathbb{R}$ and $f'(x) > f(x)$ for all $x \in \mathbb{R}$. If $f(0) = 0$, then show that $f(x) > 0$ for all $x \in \mathbb{R}$.
- ii) A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x) = 0, \forall x \in \mathbb{Q}$. Prove that $f(x) = 0, \forall x \in \mathbb{R}$.
- iii) Give an example of a function f which satisfies the intermediate-value property on a closed and bounded interval $[a, b]$, but is not continuous on $[a, b]$.

[Turn over]

- iv) Let a function $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} / \mathbb{Q} \end{cases}$$

Then show that $f'(0) = 0$.

- v) Find $\text{grad } \log |\vec{r}|$.
- vi) Give an example of a function which satisfies all the conditions of Rolle's theorem, where the derivative of the function vanishes at two distinct interior points.
- vii) Determine the constants a, b, c so that the vector $\vec{A} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.
- viii) State Rolle's theorem.
- ix) Find $\lim_{x \rightarrow 0} e^x \text{sgn}(x + [x])$, where the signum function is defined as

$$\text{sgn}(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

- x) Give an example of a function f which is continuous on a closed interval I , but $f(I)$ is not a closed interval.
- xi) If $f(x) = \max \{x, x^{-1}\}$, for $x > 0$, then obtain the value of $f(c) \cdot f(c^{-1})$ at $c > 0$.

- xii) A particle moves along the curve $x = 2t^2, y = t^2 - 4t, z = 3t - 5$. Determine the velocity and acceleration at any time $t = 1$.
- xiii) If $f(x) = -\sqrt{1-x^2}$, then show that $f'(x)$ assumes every real value as x runs over $(-1, 1)$.
- xiv) If the vectors $\vec{A}$ and $\vec{B}$ are irrotational, then show that the vector $\vec{A} \times \vec{B}$ is solenoidal.
- xv) If f is derivable on $[0, 1]$, then show by Cauchy's mean value theorem that $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one solution in $(0, 1)$.

2. Answer any **four** questions:

5 × 4 = 20

- a) i) The functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ both are continuous on $\mathbb{R}$. Then show that the set $S = \{x \in \mathbb{R}: f(x) = g(x)\}$ is a closed set in $\mathbb{R}$.
- ii) Construct a function $f: [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$ which is unbounded on the closed interval and discontinuous only at the point $\frac{\pi}{2}$. 4 + 1
- b) Let $D \subset \mathbb{R}$. A function $f: D \rightarrow \mathbb{R}$ is uniformly continuous on D if and only if for every pair of sequences $\{x_n\}$ and $\{y_n\}$ in D satisfying $\lim_{n \rightarrow \infty} |f(x_n) - f(y_n)| = 0$ holds. 5

- c) i) Let $f: [a, b] \rightarrow \mathbb{R}$ be a thrice differentiable function on $[a, b]$. If

$f(a) = f(b) = f'(a) = f'(b) = 0$, then prove that $f'''(c) = 0$ for some $c \in (a, b)$.

- ii) Show that between any two roots of $e^x \sin x = 1$, there is a real root of $e^x \cos x + 1 = 0$. 3+2

- d) A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $f(x+y) = f(x) \cdot f(y) \forall x, y \in \mathbb{R}$. If $f(1) = c$ then show that $f(x) = cx, \forall x \in \mathbb{R}$. 5

- e) i) If $f(x) = \begin{cases} 2x+3, & \text{if } x \leq 1 \\ ax^2+bx, & \text{if } x > 1 \end{cases}$

where a and b are real constants and if f is derivable everywhere, then find the values of a and b .

- ii) Show that $f(x) = [x]$ in $[0, 2]$ is not derivable of any function. 3+2

- f) If f is derivable in $[a, b]$ and $f'(a)$ and $f'(b)$ are of opposite signs, then show that there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$. 5

3. Answer any two questions:

$10 \times 2 = 20$

- a) i) If $f(x) = \sin x$, then prove that

$\lim_{h \rightarrow 0^+} \theta = \frac{1}{\sqrt{3}}$, where θ is given by

$$f(h) = f(0) + hf'(\theta h), 0 < \theta < 1.$$

- ii) If $\bar{w}$ is a constant vector, $\bar{r}$ and $\bar{s}$ are vectors functions of a scalar variable t and if $\frac{d\bar{r}}{dt} = \bar{w} \times \bar{r}$, $\frac{d\bar{s}}{dt} = \bar{w} \times \bar{s}$, then show that $\frac{d}{dt}(\bar{r} \times \bar{s}) = \bar{w} \times (\bar{r} \times \bar{s})$.

- iii) Use Cauchy's mean value theorem to evaluate

$$\lim_{h \rightarrow 1} \frac{\cos \frac{1}{2} \pi x}{\log \left(\frac{1}{x} \right)}$$

$4+3+3$

- b) i) A function $f: [0, 1] \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x, & x \text{ is rational in } [0, 1] \\ 1-x, & x \text{ is irrational in } [0, 1] \end{cases}$$

Show that (i) f is injective on $[0, 1]$, (ii) f assumes every real number in $[0, 1]$, (iii) f is continuous at $\frac{1}{2}$ and discontinuous at every other point in $[0, 1]$. 2+1+2

- ii) Show that the vector $\vec{F} = (4xy - z^3)\mathbf{i} + 2x^2\mathbf{j} - 3xz^2\mathbf{k}$ is irrotational. Show that $\vec{F}$ can be expressed as the gradient of a scalar function ϕ .
1+4
- c) i) If $c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \dots + \frac{c_n}{n+1} = 0$, where $c_i \in \mathbb{R}$ for $i = 0, 1, \dots, n$, show that the equation $c_0 + c_1x + c_2x^2 + \dots + c_nx^n = 0$ has at least one real root between 0 and 1. 4
- ii) A function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the condition $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function on $\mathbb{R}$. 3
- iii) Prove that $\text{div}[(\vec{r} \times \vec{a}) \times \vec{b}] = -2\vec{b} \cdot \vec{a}$. 3
- d) i) Obtain the Maclaurin's infinite series expansion of $(1+x)^m$, where m is any real number other than positive integer and $|x| < 1$.
- ii) If $f: [0, 1] \rightarrow \mathbb{R}$ is such that $f''(x) < 0$ in $[0, 1]$ and if $\phi(x) = f(x) + f(1-x)$ in $[0, 1]$, show that ϕ is monotone increasing in $[0, \frac{1}{2}]$ and it is monotone decreasing in $[\frac{1}{2}, 1]$.
5+5