

U.G. 3rd Semester Examination - 2024

MATHEMATICS

[MAJOR]

Course Code : MATH-MT-03

(Real Analysis)

[NEP-2020]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**The notations and symbols have their usual meanings.*1. Answer any ten questions: $2 \times 10 = 20$

- i) Show that if $f(x)$ is continuous on a compact set, then it is uniformly continuous there.
- ii) Does Rolle's theorem hold for $f(x) = 2 + (x - 2)^{2/3}$ on $[1, 3]$? Justify your answer.
- iii) Show that the function,

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ is differentiable but}$$

not continuously differentiable at $x = 0$.

- iv) Find the radius of convergence of the power series $\sum n! x^n$.
- v) Prove that the set of all limit points of a given set is closed.
- vi) Prove that a differentiable function with a bounded derivative is Lipschitz continuous.

[Turn over]

vii) Is the sequence $x_n = n(-1)^n$ bounded? Justify your answer.

viii) Show that a sequence (x_n) satisfying $|x_{n+1} - x_n| \rightarrow 0$ need not be convergent.

ix) Show that the ratio test is inconclusive when

$$\limsup \left| \frac{a_{n+1}}{a_n} \right| = 1.$$

x) Prove that every monotonic function on $[a, b]$ is Riemann integrable.

xi) Show that the alternating series test is not sufficient to guarantee the absolute convergence of a series.

xii) Prove that a bounded and monotonic sequence of partial sums must converge.

xiii) Show that the supremum of a continuous function on an open interval need not be attained.

xiv) Show that if a sequence of continuous functions $f_n(x)$ converges uniformly to $f(x)$, then $f(x)$ must be continuous.

xv) Define a convex function with an example.

2. Answer any four questions: $5 \times 4 = 20$

i) Define a Cauchy sequence with an example. Prove that a sequence (x_n) is Cauchy if and only if all of its subsequences are also Cauchy.

ii) Prove that Dirichlet function on $[0, 1]$ is discontinuous everywhere.

iii) Define functions of bounded variation on $[a, b]$. Prove that a function is of bounded variation on $[a, b]$ if and only if it can be expressed as the difference of two monotone increasing functions on $[a, b]$.

iv) Prove that the function $f(x)$ is continuous at every irrational number but discontinuous at every rational number, where

$$f(x) = \begin{cases} 1/q, & x = p/q, p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ are coprime} \\ 0 & x \text{ is irrational} \end{cases}$$

v) Consider the sequence of functions $f_n(x) = (1-x)^n$ on $[0, 1]$. I) Find pointwise limit function $f(x)$. II) Prove that $f_n(x)$ does not converge uniformly.

vi) If $f: [a, b] \rightarrow \mathbb{R}$ be differentiable, then show that f' satisfies the intermediate value property.

3. Answer any two questions: $10 \times 2 = 20$

i) a) Prove that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.
b) Provide an example of a compact metric space that is not sequentially compact, or prove that such an example is impossible.

$5 + 5$

ii) a) Let (x_n) be a bounded sequence in $\mathbb{R}$. Prove that it has convergent subsequence (Bolzano-Weierstrass theorem).
b) Give an example of a sequence that is bounded but has not limit points in $\mathbb{Q}$.

- c) Prove that if $x_n \rightarrow L$ and $y_n \rightarrow M$, then
 $\limsup (x_n + y_n) = \limsup x_n + \limsup y_n$.
 5+2+3
- iii) a) Provide an example of a Cauchy sequence in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$, and explain why.
- b) Prove that $f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous but it is not a function of bounded variation.
 3+7
- iv) a) Show that the sequence of functions $f_n(x) = \frac{x}{1+nx}$ on $[0,1]$ converges pointwise but not uniformly.
- b) Prove that the improper integral, $\int_0^1 \frac{1}{x^p} dx$ converges if and only if $p > 1$.
 5+5
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