

U.G. 5th Semester Examination - 2025**MATHEMATICS****[MAJOR-VI]****Course Code : MATH-M-T-6****(Riemann Integration and Series of Functions)****[NEP-2020]**

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

*The figures in the right-hand margin indicate marks.
The notations and symbols have their usual meanings.*

1. Answer any ten questions: $2 \times 10 = 20$ a) Prove that the function f defined on $[0,1]$ as

$$f(x) = 2n \text{ if } x = \frac{1}{n}, n = 1, 2, 3, \dots \text{ and}$$

$= 0$ otherwise is not Riemann integrable on $[0,1]$.

b) Show that $\lim_{x \rightarrow 0} \frac{x}{1-e^{x^2}} \int_0^x e^{t^2} dt = -1$.c) Examine the convergence of $\int_0^\infty \frac{\log x}{x^2} dx$.

d) State Abel's theorem in connection with the convergent of power series.

e) Show that $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$.*[Turn over]*

- f) Write the general form of a Fourier series of a function $f(x)$ defined on $(-\pi, \pi)$. State the condition under which a Fourier series contains only cosine terms.
- g) Examine the convergence of $\int_0^1 \frac{dx}{x^{\frac{1}{3}}(1+x^2)}$.
- h) Define the Riemann integral using Riemann sums.
- i) Show that both the power series $\sum_{n=0}^{\infty} a_n x^n$ and corresponding series of derivatives $\sum_{n=0}^{\infty} n a_n x^{n-1}$ have the same radius of convergence.
- j) If a function f is continuous on $[-1, 1]$, $f(x) \geq 0$ on $[-1, 1]$ and $\int_{-1}^1 f(x) dx = 0$, prove that $f = 0$ on $[-1, 1]$ identically.
- k) State Cauchy's criterion for uniform convergence of series of functions.
- l) Verify whether the sequence of functions $\{f_n(x)\}$, where $\{f_n(x)\} = x^n$, $x \in [0, 1)$ is uniformly convergent.
- m) Give example of Riemann integrable function defined on $[0, 2]$ and having discontinuity only at 1 and 2.

- n) Determine the radius of convergence of the power series

$$\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \dots$$

- o) Evaluate $\int_0^2 x[2x]dx$, where $[x]$ denotes the greatest integer function.

2. Answer any four questions: 5×4=20

- a) If f is continuous in $[a, b]$, then show that there exists a number μ , lying between a and b such that $\int_a^b f(x)dx = (b-a)f(\mu)$.
- b) State M_n test for uniform convergence of sequence of functions. Show that the sequence of functions $\{f_n(x)\}$ where $f_n(x) = (\sin nx) / \sqrt{n}$ is uniformly convergent on $[0, \pi]$ using M_n test.
- c) Find the power series for $\tan^{-1}(x)$ by integration and deduce the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$
- d) Let $\{f_n(x)\}$ be a sequence of continuous functions on $[a, b]$ which converges to f on $[a, b]$. Can you conclude that f is also continuous? Show that the sequence $\{f_n(x)\}$ where $\{f_n(x)\} = \tan^{-1} nx$ is not uniformly convergent on $[0, 1]$.

e) Show that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$.

f) Show that the series $\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \dots + \frac{x}{\{(n-1)x+1\}\{nx+1\}} + \dots$ can be shown to be non-uniformly convergent on $[0,1]$ but it still can be integrated term by term.

3. Answer any two questions: $10 \times 2 = 20$

a) i) If a function $f: [a, b] \rightarrow \mathbb{R}$ is monotonic in $[a, b]$, then show that it is integrable in $[a, b]$. Give an example of Riemann integrable function on $[0,1]$ which is not monotonic on $[0,1]$.

ii) Find the Fourier series of the periodic function f with period 2π , where $f(x) = 0$, $-\pi < x < a$; $f(x) = 1$, $a \leq x \leq b$; $f(x) = 0$, $b < x < \pi$. Find the sum of the series for $x = 4\pi + a$ and deduce that $\sum_{n=1}^{\infty} \frac{\sin n(b-a)}{n} = \frac{\pi - b + a}{2}$. $(3+2)+5$

b) i) State the second mean-value theorem of Integral Calculus in Weierstrass form and

apply it to prove that $\left| \int_{x^1}^{x^2} \frac{\sin x}{x} dx \right| < \frac{4}{x^1}$.

ii) State Dirichlet test for convergence on the integral of product. Apply it to show that $\int_0^{\infty} \sin x^2 dx$ is convergent. $5+5$

c) i) A function $f: [0, 1] \rightarrow \mathbb{R}$ is defined on $[0,1]$ by $f(0) = 0$ and

$f(x) = 0$ if x be irrational

$$= \frac{1}{q} \text{ if } x = \frac{p}{q}, \quad \text{where } p, q \text{ are}$$

positive integers prime to each other. Show that f is integrable on $[0,1]$ and $\int_0^1 f(x) dx = 0$.

ii) Show that $\int_0^{\pi/2} \cos^l x \sin^m x dx$ is convergent if and only if $l > -1, m > -1$. $5+5$

d) i) Let a function $f: [a, b] \rightarrow \mathbb{R}$ be bounded on $[a,b]$, Then show prove that f is integrable on $[a,b]$ if and only if for each $\epsilon > 0$ there exists a partition P of $[a,b]$ such that $u(P,f) - L(P,f) < \epsilon$.

ii) Show that $\int_0^{\pi/2} \log \sin x dx$ is convergent and hence evaluate it. $5+5$