

101/Math(N)

UG/1st Sem/MATH-M-T-1/25

U.G. 1st Semester Examination - 2025

MATHEMATICS

[MAJOR]

Course Code : MATH-M-T-1

(Calculus & Analytic Geometry)

[NEP-2020]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The notations and symbols have their usual meanings.

1. Answer any ten questions: $2 \times 10 = 20$

a) If $y = (x^2 - 1)^n$, then prove that

$$(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$$

b) Show that $\frac{x}{1+x} < \log(1+x) < x \quad \forall x > 0.$

c) For the equiangular spiral $r = ae^{\theta \cot \alpha}$, prove that the radius of curvature subtends a right angle at the pole.

d) Find the curvature of the curve

$$(x^2 + y^2)^2 = a^2(y^2 - x^2) \text{ at } (0, a).$$

e) Find the asymptotes of $xy^2 - y^2 - x^3 = 0.$

f) Find the equation of the parabola whose focus is the origin and whose directrix is the straight line $2x + y = 1.$

g) Does the equation $x^2 + 3xy + 2y^2 = 0$ represents a pair of straight line? If so what is the angle between them?

[Turn over]

h) Find the nature of the conic

$$\frac{5}{r} = 2(1 - \cos \theta).$$

Also find its semi-latus-rectum.

i) Find the equation of the sphere which has the points (3, 4, -1) and (-1, 2, 3) as the ends of diameter and also find its radius.

j) Find the equation of the hyperbola whose asymptotes are $2x - y = 3$ and $3x + y = 7$ and which passes through the point (1, 1).

k) Find the values of c for which the plane $x + y + z = c$ touches the sphere

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0.$$

l) To what points the origin is to be moved so that one can get rid of the first degree terms from the equation $x^2 + xy + 2y^2 - 7x - 5y + 12 = 0$?

m) Prove that $\lim_{n \rightarrow \infty} \left[\frac{n!}{n^n} \right]^{\frac{1}{n}} = \frac{1}{e}$.

n) Prove that $\int_0^{\pi/2} \frac{dx}{2 + \cos x} = \frac{\pi}{3\sqrt{3}}$.

o) Trace the curve $ay^2 = x^2(a - x)$.

2. Answer any four questions: 5 × 4 = 20

a) If the pair of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then prove that $pq = -1$.

b) Find the shortest distance between the lines $x + 4y - 2z - 5 = 0$; $x + 2y + 2z - 13 = 0$ and $2x - 2y + z - 1 = 0$; $2x + 4y - z - 9 = 0$.

c) A sphere of constant radius d passes through the origin and intersect the coordinate axes in P, Q, R . Prove that the centroid of the triangle PQR lies on the sphere $9(x^2 + y^2 + z^2) = 4d^2$.

d) If $y = \frac{1}{\sqrt{a^2 - b^2 - c^2}} \cos^{-1} \left(\frac{a\theta - a^2 + b^2 + c^2}{\theta \sqrt{b^2 + c^2}} \right)$

and $\theta = a + b \cos x + c \sin x$, prove that $\frac{dy}{dx} = \frac{1}{\theta}$.

e) Prove that, if $a > 0$, then $\lim_{x \rightarrow 0+} \frac{x}{a} \left[\frac{b}{x} \right] = \frac{b}{a}$ and

$\lim_{x \rightarrow 0+} \left[\frac{x}{a} \right] \frac{b}{x} = 0$, where $[x]$ is the greatest integer

in x but not greater than x . Discuss the left hand limits of these functions.

f) If ρ and ρ' be the radii of curvature at the ends of two conjugate diameters of an ellipse, prove that $(\rho^{2/3} + \rho'^{2/3})(ab)^{2/3} = a^2 + b^2$.

3. Answer any two questions: 10 × 2 = 20

a) i) Find $\int \sin^4 x \cos^2 x \, dx$.

ii) If $I_n = \int_0^{\pi/2} x^n \sin x \, dx$ and $n > 1$, show that

$$I_n + n(n-1)I_{n-2} = n \left(\frac{n}{2} \right)^{n-1}. \quad 5+5$$

b) i) Prove that

$$\iint \{2a^2 - 2a(x+y) - (x^2 + y^2)\} dx dy = 8\pi a^4,$$

where the region of integration being the region bounded by the circle $x^2 + y^2 + 2a(x+y) = 2a^2$.

ii) Differentiate $x^{\sin^{-1} x}$ with respect to $\sin^{-1} x$. 5+5

c) i) Show that the pole of any tangent to the hyperbola $xy = c^2$ with respect to the circle $x^2 + y^2 = a^2$ lies on a concentric and similar hyperbola.

ii) Reduce the following equation in canonical form and determine the nature of the conic:

$$x^2 - 2xy + 2y^2 - 4x - 6y + 3 = 0. \quad 5+5$$

d) i) Prove that the locus of the foot of the perpendicular from a focus of the conic $\frac{l}{r} = 1 - e \cos \theta$ on a tangent to it is given

$$\text{by } r^2(1 - e^2) - 2ler \cos \theta - l^2 = 0.$$

ii) Find the equation of the tangent planes to the ellipsoid $\frac{x^2}{144} + \frac{y^2}{36} + \frac{z^2}{9} = 1$ which are parallel to the plane $x - 2y - 2z + 1 = 0$ and determine the distance between these tangent planes. 5+5