

U.G. 2nd Semester Examination - 2024

MATHEMATICS

[HONOURS]

Generic Elective Course (GE)

Course Code : MATH-H-GE-T-02

[CBCS]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**The symbols and notations have their usual meanings.*1. Answer any ten questions: $2 \times 10 = 20$ a) Explain whether the function $f(x) = [x]$ is continuous at $x = 1$ or not.b) If $f(t)$ be odd function of t then prove that $\int_a^x f(t)dt$ is even.c) Evaluate: $\int_0^{\frac{\pi}{2}} \cos^7 x dx$.d) If $f'(x)$ exist on $[0,1]$, then show that $f(1) - f(0) = \frac{e-1}{e} f'(x)$.

e) Examine that Rolle's theorem is applicable on the function

$$f(x) = \begin{cases} 1 - x^2, & x \leq 0 \\ \cos x, & x > 0 \end{cases} \text{ in } \left[-1, \frac{\pi}{2}\right].$$

f) Applying MVT prove that $\frac{x}{1+x} < \log(1+x) < x$, for all $x > 0$.

g) Explain why L' Hospital rule is not applicable

$$\text{on } \lim_{x \rightarrow 0} \frac{\sin x}{x}.$$

[Turn Over]

h) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, find value of $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$.

i) Find $\int 5^{5^x} 5^{5^x} 5^x dx$.

j) Find the value of the limit $\lim_{x \rightarrow 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$.

k) Find the singular solution of the differential equation $x p^2 = (x - a)^2$.

l) Examine that Rolle's theorem is applicable on the function $f(x) = |x - 1|$ in $(0, 2)$.

m) If $V = \sin^{-1}\left(\sqrt{\frac{\frac{1}{x^3} + \frac{1}{y^3}}{\frac{1}{x^2} + \frac{1}{y^2}}}\right)$, then show that

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = -\frac{1}{12} \tan V.$$

n) Find the maximum value of $f(x) = 2 - |x|$.

o) Show that differential function is continuous.

2. Answer any four questions: $5 \times 4 = 20$

a) Solve: $\frac{d^3 y}{dx^3} + 3 \frac{d^2 y}{dx^2} - 4y = x e^{-2x}$.

b) If $u(x, y) = \phi(xy) + \sqrt{xy} \psi\left(\frac{y}{x}\right)$, $x \neq 0, y \neq 0$, where ϕ and ψ twice differentiable function, prove that $x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0$.

c) If $I_{m,n} = \int \sin^m x \cos^n x dx$, then show that

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{(m+n)} + \frac{(n-1)}{(m+n)} I_{m,n-2}.$$

d) If $y = x \log \left(\frac{x-1}{x+1}\right)$, prove that

$$y_n = (-1)^n (n-2)! \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n} \right].$$

e) If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x dx$, n being positive integer greater than one, then show that $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$. Hence find the value of $\int_0^{\frac{\pi}{2}} x^5 \sin x dx$.

f) Find the general and singular solution of differential equation:

$$(px^2 + y^2)(px + y) = (p+1)^2.$$

g) If $F(x, y) = 0$, then show that

$$\frac{d^2 y}{dx^2} = -\frac{(F_y)^2 F_{xx} - 2F_x F_y F_{xy} + (F_x)^2 F_{yy}}{(F_y)^3}.$$

3. Answer any two questions: $10 \times 2 = 20$

a) i) Discuss the continuity of the function at the origin, $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ 4

ii) Find the Maclaurin series expansion of $\log(1+x)$ and find the region of validity of the expansion. 6

b) i) Find the expansion of $(1+x)^n$ in a power series of x and indicate the range of validity of the expansion.

ii) If $f(x+y) = f(x) + f(y)$ for all x and y and $f(x)$ is continuous at $x=0$. Then show that $f(x)$ is continuous for all values of x .

5+5

- c) i) Show that the equation $\frac{1}{(x-1)} + \frac{2}{(x-2)} + \frac{3}{(x-3)} = 0$ has roots in (1,2) and (2,3). 5

- ii) If $y = \frac{x^3}{x^2-1}$, then prove that

$$(y_n)_0 = \begin{cases} 0, & \text{if } n \text{ is even} \\ -n!, & \text{if } n \text{ is odd} \end{cases}, n > 1. \quad 5$$

- d) i) If H be a homogeneous function in x, y, z of degree n then show that :

$$\frac{\partial}{\partial x} \left(H \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(H \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(H \frac{\partial u}{\partial z} \right) = 0, \quad ,$$

$$\text{if } u = (x^2 + y^2 + z^2)^{\frac{(n+1)}{2}}.$$

- ii) If $\phi(x)$ be a polynomial in x and λ is a real number then prove that there exists a root of $\phi'(x) + \lambda\phi(x) = 0$ between any pair of roots of $\phi(x) = 0$. 5+5