

U.G. 1st Semester Examination - 2024

MATHEMATICS

[MAJOR]

Course Code : MATH-MJ-T-1
(Calculus & Analytical Geometry)
[NEP-2020]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

*The figures in the right-hand margin indicate marks.
The notations and symbols have their usual meanings.*

1. Answer any ten questions: $2 \times 10 = 20$ a) If $y = \cosh(\sin^{-1} x)$, prove that

$$(1-x^2)y_2 - xy_1 - y = 0.$$

b) Find the radius of curvature of the curve

$$x^3 + y^3 = 3axy \text{ at the point } \left(\frac{3a}{2}, \frac{3a}{2}\right).$$

c) Find the asymptotes of the curve $x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$ which are parallel to either axis.d) Examine the curve $y^2(x^2 - 9) - x^4 = 0$ for an isolated point at the origin.e) Show that the curve $y = \sin\left(\frac{x}{a}\right)$ has a point of inflexion whenever the curve crosses the x-axis.f) Find the points on the conic $\frac{15}{r} = 3 - 4\cos\theta$ whose radius vector is 3.

[Turn over]

- g) Find the values of a and f for which the equation $ax^2 - 16xy + 4y^2 - 6x + 2fy - 12 = 0$ represents a conic having no centre.
- h) Evaluate $\int_0^{\frac{\pi}{2}} \cos^7 x dx$.
- i) Does $\lim_{x \rightarrow 0} \frac{1}{1 + e^x}$ exist? Support your answer.
- j) Find the point on the parabola $\frac{l}{r} = 1 - \cos \theta$ which has the smallest radius vector.
- k) Determine the position and nature of multiple points of the curve $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$.
- l) Find the canonical equations of the straight line passing through the points $(1, -2, 3)$ and $(3, 1, -2)$.
- m) Find the equation of the sphere having the points $(1, -2, 4)$ and $(3, 2, -1)$ as the extremities of a diameter.
- n) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x \sin^2 x dx$.
- o) Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = 1$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$.
2. Answer any four questions: 5 × 4 = 20
- a) Show that the equation $7x^2 + 60xy + 32y^2 - 14x - 60y + 7 = 0$ represents a pair of intersecting straight lines.

- b) A variable straight line intersects the line $y = 0$, $z = c$; $x = 0$, $z = -c$ and the hyperbola $xy + c^2 = 0, z = 0$. Show that the equation of the surface generated by it is $xy + (c^2 - z^2) = 0$.
- c) Find the equation of the sphere which passes through the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and has the least possible radius.
- d) If $y = e^m \sin^{-1} x$, show that $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0$.
- e) Find a and b in order that $\lim_{x \rightarrow 0} \frac{a \sin 2x - b \sin x}{x^3} = 1$.
- f) Find the asymptotes of the curve

$$x = \frac{t^2}{1+t^3}, y = \frac{t^2+2}{1+t}$$

3. Answer any two questions: 10 × 2 = 20
- a) i) Prove that the asymptotes of the cubic $(x^2 - y^2)y - 2ay^2 + 5x - 7 = 0$ form a triangle of area a^2 .
- ii) Evaluate $\int \frac{\sin^3 x}{\cos^4 x} dx$ 5+5
- b) i) Find the equations of the lines in which the plane $2x - y + z = 0$ cuts the cone $8x^2 - 2y^2 + z^2 = 0$.
- ii) Find $\frac{dy}{dx}$ when $y = x^{\tan x} + (\sin x)^{\cos x} + 10x^2$. 5+5

- c) i) Show that the plane $x+3y+6z=27$ is a tangent plane to the ellipsoid $\frac{x^2}{81} + \frac{y^2}{36} + \frac{z^2}{9} = 1$ and find the coordinates of the point of contact.

- ii) Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters are connected by $a^2 + b^2 = c^2$. 5+5

- d) i) A conic $\frac{l}{r} = 1 + e \cos \theta$ is cut by a circle passing through the pole in four points whose radius vectors are r_1, r_2, r_3 , and r_4 .

Show that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}$.

- ii) Prove that $\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{(n-1)(n-3)\dots 4.2}{n(n-1)\dots 3.1}$, if n be an odd integer, and $n > 1$. 5+5
-