

U.G. 4th Semester Examination - 2024

MATHEMATICS**[HONOURS]****Generic Elective Course (GE)****Course Code : MATH-H-GE-T-04****(Calculus & Differential Equations)**

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

*The figures in the right-hand margin indicate marks.
The symbols and notations have their usual meanings.*

1. Answer any ten questions: $2 \times 10 = 20$
- a) Obtain the differential equation of all circles each of which touches the axis of x at the origin.
- b) Verify whether $(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$ is exact or not.
- c) Obtain the singular solution of the differential equation $y = px + \sqrt{1 + p^2}$.
- d) Show that $\frac{1}{3x^3y^3}$ is an integrating factor of $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$.
- e) Eliminate the constants A and B from the relation $v = \frac{A}{r} + B$ to construct a differential equation of second order.

[Turn Over]

f) If $y = a \cos(\log x) + b \sin(\log x)$, show that $x^2 y^2 + xy_1 + y = 0$.

g) Find $\int_0^{\frac{\pi}{2}} \sin^5 x dx$.

h) A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \frac{1}{2} - x & \text{when } 0 < x < \frac{1}{2} \\ \frac{1}{2} & \text{when } x = \frac{1}{2} \\ \frac{3}{2} - x & \text{when } \frac{1}{2} < x < 1 \end{cases}$$

Show that $f(x)$ is discontinuous at $x = \frac{1}{2}$.

i) A function is defined in the following way

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that $f'(0)$ does not exist.

j) Evaluate $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}$.

k) Find the value of $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$.

l) Find the value of 'a' so that $\left(\frac{1}{x} + x^a y^4\right) dx = y^3 x^{a+1} dy$ may be exact.

m) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n \theta d\theta$, then find the value of $I_0 + I_2$.

n) If $f(x, y) = \tan^{-1} \frac{y}{x} + \sin^{-1} \frac{x}{y}$, find the value of

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$$

2. Answer any **four** questions: 5 × 4 = 20

a) Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$ for $0 < x < \frac{\pi}{2}$.

b) If $V = 2 \cos^{-1} \left(\frac{x+y}{\sqrt{x} + \sqrt{y}} \right)$, show that

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + \cot \frac{V}{2} = 0.$$

c) Evaluate $\lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^{\frac{1}{x}}$.

d) Find reduction formulae for $\int \sin^n x dx$, n being positive integer greater than 1.

e) Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$.

f) Solve by the method of variation of parameters

$$\frac{d^2 y}{dx^2} + a^2 y = \tan ax.$$

3. Answer any two questions:

10×2=20

a) i) Show that $x^{\frac{1}{x}}$ ($x > 0$) is maximum at $x = e$.

ii) If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, show that

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}} \quad 5+5$$

b) i) Solve and find the singular solution of differential equation

$$\sin\left(x \frac{dy}{dx}\right) \cos y = \cos\left(x \frac{dy}{dx}\right) \sin y + \frac{dy}{dx}$$

ii) Verify Rolle's theorem for the function

$$f(x) = x^2 - 5x + 6 \text{ in } 1 \leq x \leq 4. \quad 5+5$$

c) i) If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}, |x| < 1$, show that

$$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0.$$

ii) Find reduction formulae for

$$\int \sin^m x \cos^n x dx, \text{ where } m \text{ \& } n \text{ are positive integers, greater than 1.}$$

5+5