

U.G. 5th Semester Examination - 2024

MATHEMATICS**[HONOURS]****Discipline Specific Elective (DSE)****Course Code : MATH-H-DSE-T-2B****(Differential Geometry)****[New Syllabus]**

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**Symbols and notations have their usual meanings.*

1. Answer any ten questions: $2 \times 10 = 20$
- a) Verify whether the curve $\gamma(t) = (\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}})$ is a unit speed curve.
- b) Define a regular curve. Verify whether the curve $\gamma(t) = (\cos^2 t, \sin^2 t)$ for $-\infty < t < \infty$ is regular.
- c) Show that the curvature of a circle is inversely proportional to its radius.
- d) Define signed curvature of a unit speed curve. What is the signed curvature of the curve $\gamma(s) = (R \cos \frac{s}{R}, R \sin \frac{s}{R})$, where R is a constant.

[Turn over]

- e) Let $y(t)$ be a unit speed curve in $\mathbb{R}^3$, and assume that its curvature $k(t)$ is non-zero for all t . Define a new curve σ by $\sigma(t) = \frac{dy(t)}{dt}$. Find the curvature of σ .
- f) Define simple closed curve. Is the curve $y(t) = (\cos \frac{2\pi t}{a}, \sin \frac{2\pi t}{a})$ simple closed? If so find its interior and exterior.
- g) Define a surface. Is an open disc in the xy -plane a surface? Support your answer.
- h) Define an orientable surface. Give an example of a surface which is not orientable.
- i) When two surfaces are called isometric? Give example of two surfaces which are isometric.
- j) Calculate the first fundamental form of the surface given by $\sigma(u,v) = (u-v, 2v, u)$.
- k) Can you change the first fundamental form of a surface by rigid motion? Support your answer.
- l) When a diffeomorphism between two surfaces is called conformal? Give an example of conformal map between two surfaces.
- m) Prove that every isometry is conformal.

- n) Prove that principal curvatures of a surface at one of its points are roots of the equation $x^2 - 2Hx + K = 0$, where H and K are respectively the mean and Gaussian curvatures of the surface at the point.
- o) Prove that any normal section of a surface is a geodesic.

2. Answer any four questions: 5×4=20

- a) Prove that a parametrized curve has unit speed reparametrization if and only if it is regular.
- b) State and prove Serret-Frenet formula for regular curves in $\mathbb{R}^3$.
- c) Prove that every simple closed curve in $\mathbb{R}^2$ has at least four vertices.
- d) Let U and U' be open subsets of $\mathbb{R}^2$ and let $\sigma: U \rightarrow \mathbb{R}^3$ be a regular surface patch. Let $\mu: U' \rightarrow U$ be bijective smooth map with smooth inverse map $\mu^{-1}: U \rightarrow U'$. Prove that $\sigma' = \sigma \circ \mu: U' \rightarrow \mathbb{R}^3$ is a regular surface patch.
- e) Let $\phi: U \rightarrow \mathbb{R}^3$ be a patch of a surface S containing a point P of S , and let (u,v) be coordinates in U . Show that the tangent space of S at P is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v .

f) Find the Gaussian curvature of the unit sphere.

3. Answer any two questions: $10 \times 2 = 20$

a) Let $y(t)$ be a unit speed curve in R^3 and assume that its curvature $k(t)$ is non-zero for all t and τ

denotes its torsion. Let $\delta(t) = \frac{dy(t)}{dt}$. Show that

δ is regular and that, if s is arc length parameter

for δ , then $\frac{ds}{dt} = k$. Prove that curvature of δ is

$\left(1 + \frac{\tau^2}{k^2}\right)^{\frac{1}{2}}$. Also find the torsion of δ in terms

of κ and τ .

$2+1+3+4$

b) Define Gaussian curvature and mean curvature of a surface. Find the Gaussian curvature and mean curvature of the surface given by $\alpha(u, v) = (u, v, uv)$.

$2+4+4$

c) Let γ be a curve on a surface patch σ , and let κ_1 and κ_2 be the principal curvatures of σ , with non-zero principal vectors t_1 and t_2 . Show that the normal curvature of the curve is $\kappa_n = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta$, where θ is the angle between $\dot{\gamma}$ and t_1 .

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