

U.G. 3rd Semester Examination - 2024**MATHEMATICS****[HONOURS]****Generic Elective Course (GE)****Course Code : MATH-H-GE-T-03****(Algebra & Analytical Geometry)****[CBCS]**

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**Symbols and notations have their usual meanings.*

1. Answer any ten questions: $2 \times 10 = 20$
- a) For a non-singular matrix A , prove that $|A^{-1}| = |A|^{-1}$.
- b) Verify $\langle A, + \rangle$ form a group or not where $A = \{-1, 0, 1\}$.
- c) Give an example of a relation which is reflexive, transitive but not symmetric.
- d) Given $f: \mathbb{N} \rightarrow \mathbb{N}$, where $f(x) = x - (-1)^2$, $x \in \mathbb{N}$ where $\mathbb{N}$ is a set of natural numbers. Verify whether f is bijective or not.

[Turn over]

e) Find the angle between the pair of straight lines $3x^2 - 10xy + 3y^2 = 0$.

f) Diminish the roots of the equation $x^4 - 5x^3 + 7x^2 - 17x + 11 = 0$ by 4.

g) For the curve $3x^2 - 4xy + 2y^2 + 4x - 2y + 1 = 0$, determine whether it has a single centre, has no centre or has infinitely many centres.

h) Determine the rank of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & -4 \end{bmatrix}$

i) In a group $\langle G, o \rangle$, the order of the distinct element a, b is 2. If a, b commute, then find the order of $a o b$.

j) Let $(G, .)$ be a group and $a \in G$, such that $a^2 = a$. Then show that a is the identity element of G .

k) a and b are any two elements in a group $\langle G, . \rangle$. If $(a, b)^2 = a^2 b^2$, show that the group is abelian.

l) Find the nature of the conic $x^2 - 2xy + 2y^2 - 4x - 6y + 3 = 0$.

m) Find the polar equation of the straight line joining the two points $\left(1, \frac{\pi}{2}\right)$ and $(2, \pi)$.

n) If $|A|=2$ and $\text{Adj } A = \begin{bmatrix} -2 & 3 & 1 \\ 6 & -8 & -2 \\ -4 & 7 & 1 \end{bmatrix}$, then find A .

o) Transform the equation $5x^2 - 4xy + 5y^2 = 21$ to axes inclined at an angle 45° to the original axes.

2. Answer any four questions: 5×4=20

a) If $\langle G, . \rangle$ be an abelian group with identity element e , then prove that all elements $x \in G$ satisfying the equation $x^2 = e$ form a sub-group of G .

b) In the symmetric group S_3 , show that the subsets $A = \{e, (1, 2)\}$ and $B = \{e, (1, 2, 3), (1, 3, 2)\}$ are subgroups. Also show that $A \cup B$ is not a subgroup of S_3 .

c) If α, β, γ be the roots of the equation $x^3 + 3x^2 - x - 1 = 0$, then find the equation whose roots are

$$\frac{\alpha}{\beta + \gamma}, \frac{\beta}{\gamma + \alpha}, \frac{\gamma}{\alpha + \beta}.$$

Hence find $\sum \frac{\alpha}{\beta + \gamma}$.

- d) Prove that the equation to the locus of the foot of the perpendicular from the focus of the conic

$$\frac{l}{r} = 1 + e \cos \theta \text{ on a tangent to it is the circle}$$

$$r^2(e^2 - 1) - 2ler \cos \theta + l^2 = 0.$$

- e) Find the minimum number of imaginary roots of the equation $4x^7 - 8x^4 + 4x^3 - 7 = 0$.

- f) Prove that the pair of straight lines joining the origin to the points of intersection of the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ by the straight line } lx + my + n = 0$$

are coincident if $a^2l^2 + b^2m^2 = n^2$.

- g) Solve completely the following system of equations, if possible:

$$x + 2y + 3z = 0$$

$$2x + 3y + 4z = 0$$

$$3x + 4y + 5z = 0$$

- h) If $\langle G, \cdot \rangle$ is a group and $a^{-1}b^2(bab)^{-1} = b$ for all $a, b \in G$ then show that G is a commutative group.

3. Answer any two questions:

$$10 \times 2 = 20$$

- a) i) Reduce the matrix A to a row reduced echelon form and hence find its rank, where

$$A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}.$$

- ii) Investigate for what value of λ and μ , the following system of equations

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

have (I) no solution (II) unique solution and (III) an infinite number of solutions.

5+5

- b) i) If the lines $ax^2 - 2hxy + by^2 = 0$ form an equilateral triangle with the line

$$x \cos \alpha + y \sin \alpha = p, \text{ show that}$$

$$\frac{a}{1 - 2 \cos 2\alpha} = \frac{h}{2 \sin 2\alpha} = \frac{b}{1 + 2 \cos 2\alpha}$$

- ii) Find the angle of rotation of the axes for which the equation $x^2 - y^2 = a^2$ reduces to $xy = c^2$. Determine c^2 . 5+5

- c) i) Prove that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

where n is an integer.

- ii) If

$$x + \frac{1}{x} = 2 \cos \alpha, y + \frac{1}{y} = 2 \cos \beta, z + \frac{1}{z} = 2 \cos \gamma$$

then prove that $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$
if $x + y + z = 0$.

- d) i) Solve the equation $x^3 - 3x^2 + 12x + 16 = 0$ by Cardan's method.

- ii) Find the multiple root, if any, of the equation $x^3 + x^2 - 16x + 20 = 0$ and hence solve it.