

## U.G. 4th Semester Examination - 2024

## MATHEMATICS

## [HONOURS]

Course Code : MATH-H-CC-T-8

(Ring Theory and Linear Algebra)

Full Marks : 60

Time :  $2\frac{1}{2}$  Hours

*The figures in the right-hand margin indicate marks.  
The symbols and notations have their usual meanings.*

1. Answer any ten questions:  $2 \times 10 = 20$ 
  - a) Suppose that  $R$  is a ring such that  $x^3 = x$  for all  $x$  in  $R$ . Prove that  $6x = 0$  for all  $x$  in  $R$ .
  - b) Find a basis of the subspace
 
$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2 \times 2} : a + b = 0 \right\}$$
 of the vector space  $\mathbb{R}_{2 \times 2}$ .
  - c) Is  $\mathbb{Z}_6$  a subring of  $\mathbb{Z}_{12}$ ?
  - d) Does product of two diagonalizable matrices diagonalizable? If not, then give an example.
  - e) Find the maximal and the prime ideals in the ring  $\mathbb{Z}_6$ .
  - f) Show that  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by
 
$$T(x, y, z) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, z)$$
 is non-singular.
  - g) Show that  $2\mathbb{Z} \cup 3\mathbb{Z}$  is not a subring of  $\mathbb{Z}$ .
  - h) Let  $P$  is a real orthogonal matrix with  $\det P = -1$ . Prove that  $-1$  is an eigen value of  $P$ .

[Turn Over]

i) Determine all elements of a ring are both units and idempotents.

j) Let  $V$  be the vector space of all functions  $f$  from  $\mathbb{R}$  into  $\mathbb{R}$ , and

$W = \{f : f \in V \text{ and } f(x^2) = [f(x)]^2\}$ . Is  $W$  a subspace of  $V$ ?

k) Show that  $\frac{\mathbb{Z}_3[x]}{\langle x^2 + x + 1 \rangle}$  is not a field.

l) Let,  $T : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$  is a linear transformation defined as  $T(A) = \text{trace } A \cdot I_{2 \times 2} + \text{trace } A^T \cdot I_{2 \times 2}$ . What is the rank of  $T$ ?

m) Is  $W = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0\}$  a subspace of the vector space  $\mathbb{R}^3(\mathbb{R})$ ?

n) The eigen values of  $3 \times 3$  matrix  $A$  are 3, 2 and -1. If  $B = A^2 - A$  then find the determinant of  $B$ .

o) Prove that every maximal ideal in a commutative ring with unity is a prime ideal.

2. Answer any four questions: 5 × 4 = 20

a) Using Extension theorem extend the set

$S = \{(1, 1, 0, 0), (1, 1, 1, 0)\}$  to a basis of  $\mathbb{R}^4$ .

b) Write the Cayley-Hamilton theorem for a square matrix. Verify Cayley-Hamilton theorem for the

matrix  $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$ . Express  $A^{-1}$  as a polynomial in  $A$  and then compute  $A^{-1}$ .

c) The matrix of a linear mapping  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  with respect to the ordered basis

$((0, 1, 1), (1, 0, 1), (1, 1, 0))$  of  $\mathbb{R}^3$  is  $\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}$ .

Find  $T$ . Find the matrix of  $T$  relative to the standard basis of  $\mathbb{R}^3$ .

d) Find the eigen values and eigen vectors of the

matrix  $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ . Is the matrix

diagonalizable?

e) Let  $R$  be the ring of all real valued continuous functions defined on  $[0, 1]$  and let

$S = \left\{ f \in R : f\left(\frac{1}{2}\right) = 0 \right\}$ . Then show that  $S$  is an

ideal of  $R$ .

Is it prime or maximal ideal? Justify your answer.

f) i) If  $c(x) = (x-3)^3(x-2)^4$  and

$m(x) = (x-3)^2(x-2)^2$ , be the characteristic and minimal polynomial of a matrix, then find all possible Jordan Canonical forms of the matrix.

ii) Prove that the ring of real valued continuous function on  $[0, 1]$  is not an integral domain. 3+2

3. Answer any two questions: 10×2=20

a) i) Use Gram Schmidt process to find an orthonormal basis of the subspace of  $\mathbb{R}^4$  with standard inner product generated by the linearly independent set  $\{(1,1,0,1), (1,1,0,0), (0,1,0,1)\}$ .

ii) A linear mapping  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is defined by  $T(x, y, z) = (x - y, x + 2y, y + 3z)$ ,  $(x, y, z) \in \mathbb{R}^3$ . Show that  $T$  is non singular and determine  $T^{-1}$ . 5+5

b) i) Let  $V$  be a vector space of dimension 'n' over a field  $F$ . Then prove that  $V$  is isomorphic to  $F^n$ .

ii) Determine the linear mapping  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  that maps the basis vectors  $(1,0,0), (0,1,0), (0,0,1)$  of  $\mathbb{R}^3$  to the vectors  $(0,1,0), (0,0,1), (1,0,0)$  respectively. Find  $\text{Ker } T$  and  $\text{Im } T$ . Verify that,  $\dim \text{Ker } T + \dim \text{Im } T = 3$ . 5+5

c) i) Find all ring homomorphisms from  $\mathbb{Z}_{18}$  to  $\mathbb{Z}_{30}$ .

ii) State and prove the fundamental theorem of ring homomorphism. 4+1+5