

U.G. 5th Semester Examination - 2024

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-11

(Riemann Integration and Series of Functions)

[New Syllabus]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

1. Answer any ten questions: 2×10=20
- a) Let $f(x)$ be integrable on $[a, b]$. Is $f(x)$ necessarily continuous there? Give an example.
- b) Show that the function $f(x) = [x]$ is Riemann integrable in any closed interval $[a, b]$.
- c) Show that $\int_0^{\pi/2} \sqrt{\tan x} \, dx = \frac{\pi}{\sqrt{2}}$.
- d) Evaluate $\lim_{x \rightarrow 0} \frac{1}{x^2} \int_0^{x^2} e^{\sqrt{1+t}} \, dt$.
- e) Prove that $\int_0^{\infty} \frac{\cos x}{\sqrt{1+x^3}} \, dx$ converges absolutely.

[Turn over]

f) Determine the radius of convergence of the

power series $\sum_{n=1}^{\infty} \left(n^{\frac{1}{3}} + 1 \right)^n x^n$.

g) Prove that $\frac{1}{\sqrt[3]{2}} < \int_0^1 \frac{x^2}{\sqrt{1+x}} dx < \frac{1}{3}$.

h) State the fundamental theorem of integral calculus.

i) Show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is uniformly convergent for all x .

j) State Dirichlet's conditions for Fourier series expansion.

k) Prove that $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$.

l) Find a series of cosines of multiples of x which represent $f(x) = x$ on $0 \leq x \leq \pi$.

m) Show that $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$.

n) Prove that $\int_1^{\infty} \frac{x^{m-1}}{1+x} dx$ converges for $m > 0$.

o) Verify the interval of convergence of the series $1 + x + x^2 + \dots$ for $-1 < x < 1$.

2. Answer any **four** questions:

5×4=20

a) State and prove Cauchy-Hadamard formula on power series.

b) Show that $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$ exists if and only if $n > 0$.

c) Show that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

is not uniformly convergent in any interval containing zero.

d) Prove that any bounded function which is continuous except for a finite number of discontinuities is integrable.

e) Obtain the Fourier series expansion of $f(x) = x$ on $[-\pi, \pi]$ and show that

$$x = 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right].$$

Hence deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

f) If $f: [a, b] \rightarrow \mathbb{R}$ is a Riemann integrable function, then show that it is bounded. Give an example to prove that the converse is not true.

3. Answer any two questions:

10×2=20

a) i) State and prove Darboux's theorem on Riemann integral. 4

ii) Show that the function f defined on $[0, 1]$

by $f(0)=0$ and $f(x)=\frac{1}{2^n}$ for

$\frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} (n \in \mathbb{N})$ is integrable there,

although it has an infinite number of

discontinuities. Also find $\int_0^1 f(x) dx$. 4+2

b) i) State and prove Weierstrass's M-test for uniform and absolute convergence of a series of real functions. 5

ii) Prove that

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}, \quad 0 < m < 1. \quad 5$$

c) i) State and prove Abel's test for improper integrals. 5

ii) If the series $\sum_{n=1}^{\infty} u_n(x)$ converges to $S(x)$

on $a \leq x \leq b$ and if each term $u_n(x)$ is continuous on $[a, b]$, then show that $S(x)$

is continuous on $[a, b]$. 5