

U.G. 3rd Semester Examination - 2025

MATHEMATICS

[MAJOR]

Course Code : MATH-MT-03

(Real Analysis)

[NEP-2020]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**The notations and symbols have their usual meanings.*1. Answer any ten questions: $2 \times 10 = 20$

i) If $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and f is continuous at a point of $\mathbb{R}$, prove that f is uniformly continuous on $\mathbb{R}$.

ii) Find the derived set of the set $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.

iii) Is every bounded sequence convergent? Justify your answer.

iv) Prove that the function,

$$f(x) = \begin{cases} x^2 \cos(1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
 is differentiable on $\mathbb{R}$.

v) State the Sandwich theorem for sequences.

vi) Using the Mean Value Theorem, prove that

$$0 < \frac{1}{x} \log \left(\frac{e^x - 1}{x} \right) < 1, \text{ for } x > 0.$$

[Turn over]

- vii) Find the limit of the sequence $\{u_n\}$, defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ for $n \geq 1$.
- viii) Prove that the union of two open sets in $\mathbb{R}$ is an open set.

- ix) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$.
- x) Prove that the function $f : (0,1) \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous.

- xi) Prove that the Cauchy's root test for a series $\sum_{n=1}^{\infty} u_n$ is inconclusive when $\lim_{n \rightarrow \infty} u_n^{1/n} = 1$.

- xii) Prove that every Lipschitz function is uniformly continuous.

- xiii) Find the supremum and infimum of the set $A = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$.

- xiv) Test the convergence of the series $1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \dots, x > 0$.

- xv) Does a function that satisfies the Intermediate-value property on $[a, b]$ have to be continuous on $[a, b]$? Justify your answer.

2. Answer any **four** questions: 5×4=20

- i) Prove that the intersection of an arbitrary family of closed sets is a closed set. Is the union of an arbitrary family of closed sets always closed? Justify your answer. 3+2

- ii) Prove that a continuous function on a closed and bounded interval is uniformly continuous. 5
- iii) Define a Cauchy sequence and prove that every Cauchy sequence is a convergent sequence. 1+4

- iv) Prove that the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges, where $\{u_n\}$ is a monotone decreasing sequence of positive real numbers with $\lim_{n \rightarrow \infty} u_n = 0$. 5

- v) Define the limit point of a set $S \subset \mathbb{R}$. Show that the set of integers $\mathbb{Z}$ has no limit point. 1+4

- vi) State the Monotone convergence theorem. Show that the sequence $\{x_n\}$, defined by $x_1 = 1$; $x_{n+1} = \sqrt{2 + x_n}$, $\forall n \in \mathbb{N}$, is convergent. Also, find $\lim_{n \rightarrow \infty} x_n$. 1+4

3. Answer any **two** questions: 10×2=20

- i) a) State the Cauchy convergence criterion for sequences and hence prove that the sequence $\{x_n\}$ defined by $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ does not converge. 1+4

- b) Test the convergence of the following series:

i) $\sum_{n=1}^{\infty} \frac{1}{n^p}, p \in \mathbb{R}$

ii) $\sum_{n=1}^{\infty} \frac{1}{1+2^n}$

3+2

- ii) a) Show that the function $f: \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ -1 & \text{if } x \text{ is irrational,} \end{cases}$ is discontinuous at every point of $\mathbb{R}$. 5
- b) State and prove the Lagrange mean value theorem. 5
- iii) a) Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{7 + x_n}$ for all $n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$. 5
- b) Prove that if $a_0, a_1, a_2, \dots, a_n$ are real numbers such that $\frac{a_0}{n+1} + \frac{a_1}{n} + \dots + \frac{a_{n-1}}{2} + a_n = 0$, then there exists at least one real x between 0 and 1 such that $a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$. 5
- iv) a) If the sequence $\{x_n\}$ converges to l , prove that $\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = l$. Is the converse true? Justify your answer. 5
- b) State and prove the Bolzano-Weierstrass theorem for sets. 5
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