

U.G. 2nd Semester Examination - 2023

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-03

(Real Analysis)

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**The symbols and notations have their usual meanings.*

1. Answer any ten questions: $2 \times 10 = 20$
- a) Prove that the set of all open intervals having rational end points is a countable set.
 - b) Using the definition of a compact set, prove that a finite subset of $\mathbb{R}$ is a compact set in $\mathbb{R}$.
 - c) Let $A = \{x \in \mathbb{R} : x^2 - 5x + 6 < 0\}$. Find the *lub* and *glb* of A .
 - d) Show that the union of an infinite number of closed sets in $\mathbb{R}$ is not, necessarily a closed set.
 - e) Prove or disprove $(A \cap B)' = A' \cap B'$ and $(A \cup B)' = A' \cup B'$, where A' is the derived set of the set A .

[Turn over]

- f) If S be a non-empty compact subset of $\mathbb{R}$, then prove that $\sup S$ and $\inf S$ belong to S .
- g) Prove that $\lim r^n = 0, [r] < 1$.
- h) Three subsequences $\{x_{3n}\}, \{x_{3n-1}\}$, and $\{x_{3n-2}\}$ of a sequence $\{x_n\}$ converge to the same limit x , then prove that the sequence $\{x_n\}$ is also converges to the same limit x .
- i) Let $S \subseteq \mathbb{R}$ and x be a limit point of S . Then show that every neighbourhood of x contains infinitely many elements of S .
- j) Prove that a Cauchy sequence of real numbers is convergent.
- k) Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in $\mathbb{R}$, then show that $\{x_n y_n\}$ is also a Cauchy sequence.
- l) Prove that a bounded sequence $\{x_n\}$ is convergent if and only if $\overline{\lim} x_n = \underline{\lim} x_n$.
- m) Prove that
- $$\lim_{n \rightarrow \infty} \frac{1}{n} \{(a+1)(a+2)\dots(a+n)\}^{\frac{1}{n}} = \frac{1}{e} \text{ if } a > 0.$$
- n) State and prove the necessary condition for the convergence of a series of positive terms. Verify whether it is also sufficient condition or not.

- o) Test the convergence of the series

$$1 + \frac{1}{1!} + \frac{2^2}{2!} + \frac{3^3}{3!} + \dots$$

2. Answer any four questions: 5×4=20

- a) Prove that derived the set A' of a bounded set $A \subset \mathbb{R}$ is a compact set in $\mathbb{R}$.
- b) Find the derived set of the set $A = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \right\}$.
- c) Using Archimedean property to show that the set $\mathbb{Z}$ of intergers is neither bounded above nor bounded below.
- d) If A be a compact set in $\mathbb{R}$, then show that every infinite subset of A has a limit point in A .
- e) Prove that the set of real numbers $\mathbb{R}$ is not countable.
- f) If $\lim_{n \rightarrow \infty} x_n = a$, then prove that $\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = a$. Is the converse true? Justify your answer.
- g) Using Gauss's test to show that the series $\left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right)^p + \dots$ converges for $p > 2$ and diverges for $p \leq 2$.

3. Answer any two questions:

$10 \times 2 = 20$

- a) i) Let $A \subset \mathbb{R}$. Prove that $(A^\circ)^c = \overline{A^c}$ and $(A^c)^c = (\overline{A})^\circ$, where $A^\circ, \overline{A}, A^c$ are the interior, closure, and complement of the set A respectively.
- ii) Let A, B be two disjoint compact subsets of $\mathbb{R}$, then show that $d(A, B) > 0$, where $d(a, b) = \inf \{|a - b| : a \in A, b \in B\}$. 5+5
- b) i) State and prove Cantor's theorem on nested intervals in $\mathbb{R}$.
- ii) Let $0 < y_1 < x_1$. Define

$$x_{n+1} = \frac{x_n + y_n}{2} \text{ and } y_{n+1} = \sqrt{x_n y_n}.$$

Prove that the sequence $\{y_n\}$ is monotonic increasing and the sequence $\{x_n\}$ is monotonic decreasing but both the sequences converge to the same limit. 5+5

- c) i) State Cauchy's condensation test and use it to discuss the convergence of the

series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, p > 0$.

- ii) Show that the series

$$\frac{1}{(1+a)^r} - \frac{1}{(2+a)^r} + \frac{r}{(3+a)^r} - \dots \text{ is}$$

absolutely convergent for $p > 1$ and conditionally convergent for $0 < p \leq 1$.

5+5