

U.G. 3rd Semester Examination - 2024

MATHEMATICS**[PROGRAMME]****Course Code : MATH-G-CC-T-03****(Real Analysis)****[CBCS]**

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**Symbols and notations have their usual meanings.*

1. Answer any ten questions:

 $2 \times 10 = 20$

- a) Prove that the sequence $\left\{\frac{1}{n^p}\right\}$, where $p > 0$, is a null sequence.
- b) State Cauchy general principle of converges of a series.
- c) State Leibnitz's Test on alternating series.
- d) Show that the series $\sum u_n$ is conditionally convergent, where $u_n = (-1)^n \frac{1}{2n-3}$.
- e) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n} x^n$.

[Turn over]

- f) Show that the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.
- g) Prove that if a series $\sum u_n$ converges, then $\lim_{n \rightarrow \infty} u_n = 0$.
- h) Show that the series $\frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \dots$ is convergent.
- i) State Bolzano-Weierstrass theorem for sets.
- j) Test for the convergence of the sequence $\{x_n\}$, where $x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$, $n \in \mathbb{N}$.
- k) Define "conditional convergence" with an example.
- l) Write the completeness properties of the system $\mathbb{R}$ of real numbers.
- m) Using Cauchy's General Principle for convergence show that harmonic series $\sum u_n = \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots \infty$ does not converge.
- n) With suitable examples show that the union of an arbitrary family of closed sets may or may not be a closed set.
- o) If $\sum a_n$ be a positive convergent series, then prove that $\sum a_n^2$ also converges.

2. Answer any four questions: 5×4=20
- a) Prove that if a sequence $\{x_n\}$ of real numbers is monotonic increasing and bounded above, then it converges to its exact upper bound.
- b) Find the interval of convergence of the power series $10x + \frac{10^2}{2}x^2 + \frac{10^3}{3}x^3 + \dots + \frac{10^n}{n}x^n + \dots \infty$.
- c) Prove that if S is a closed subset of $\mathbb{R}$ then its complement S^c is open in $\mathbb{R}$.
- d) State and prove Comparison test for an infinite series of positive terms.
- e) State and prove Archimedean property of the set of real numbers $\mathbb{R}$.
- f) Find the interval of convergence of the power series $x + \frac{x^2}{2 \cdot 10} + \frac{x^3}{3 \cdot 10^2} + \dots + \frac{x^n}{n \cdot 10^{n-1}} + \dots \infty$.
- g) Show that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$.
3. Answer any two questions: 10×2=20
- a) i) Show that the power series of $\tan^{-1}x$ is given by $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \infty$ ($-1 \leq x \leq 1$).
Hence derive that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \infty$.

ii) Show that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \right) = 1.$$

(5+2)+3

b) i) Let $\{x_n\}$ be a sequence of positive real numbers defined by

$$x_{n+1} = \sqrt{7 + x_n}, \quad n \in \mathbb{N} \quad \text{and} \quad x_1 = \sqrt{7}.$$

Show that the sequence $\{x_n\}$ converges to a real root of the quadratic equation $x^2 - x - 7 = 0$. Find also the limit of the sequence.

ii) Show that $\left\{ \frac{\cos n}{n} \right\}$ is a null sequence.

(5+2)+3

c) i) Prove that every convergent sequence is bounded. Is the converse of this theorem true? Justify your answer.

ii) State and prove Sandwich theorem on sequence.

(3+2)+5

d) i) Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{6}$ and $x_{n+1} = \sqrt{6 + x_n}$ for $n \geq 1$, converges to 3.

ii) Show that $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$ by using Cauchy's second theorem on limit and hence show that $\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + 2^{\frac{1}{2}} + 3^{\frac{1}{3}} + \dots + n^{\frac{1}{n}} \right) = 1$.

5+(3+2)